Key for the Calculus Challenge Exam in 00-2

1. (a) We can divide out $x-1$ from both numerator and denominator obtaining:

$$\lim_{x\to 1}\frac{x^3-1}{x^4-1}=\lim_{x\to 1}\frac{x^2+x+1}{x^3+x^2+x+1}=\frac{\lim_{x\to 1}(x^2+x+1)}{\lim_{x\to 1}(x^3+x^2+x+1)}=\frac{3}{4}$$

One can use l'Hospital's rule to reach the same conclusion.

(b) $\displaystyle\lim_{x\to 0+}\frac{x+2}{2+\sqrt{x}}=\frac{\lim_{x\to 0+}(x+2)}{\lim_{x\to 0+}(2+\sqrt{x})}=\frac{(\lim_{x\to 0+}x)+2)}{2+\lim_{x\to 0+}\sqrt{x}}=\frac{0+2}{2+0}=1.$

(c) This is a more subtle problem; if the absolute value signs are removed, then the limit does not exist. Note that $\sqrt{1-\cos 2x}/|x|$ is an even function so it is sufficient to evaluate $\displaystyle\lim_{x\to 0+}\left(\sqrt{1-\cos 2x}/x\right)$. By l'Hospital's rule we have

$$\lim_{x\to 0+}\left(\sqrt{1-\cos 2x}/x\right)^2=\lim_{x\to 0+}\frac{1-\cos 2x}{x^2}=\lim_{x\to 0+}\frac{\sin 2x}{x}=2.$$

Hence $\lim_{x\to 0+}\sqrt{1-\cos 2x}/x=\sqrt{2}$.
Another approach is to use the identity $1-\cos 2x=2\sin^2 x$ which gives

$$\lim_{x\to 0}\frac{\sqrt{1-\cos 2x}}{|x|}=\lim_{x\to 0}\frac{\sqrt{2\sin^2 x}}{|x|}=\lim_{x\to 0}\frac{\sqrt{2}\sin x}{x}=\sqrt{2}.$$

2. The relevant theorem, the IVT, says that, if the function $f(x)$ is continuous on the closed interval $[a,b]$ of $\mathbb{R}$, and

$$f(a)<r<f(b)\ \text{ or }\ f(b)<r<f(a),$$

then there exists $c\in(a,b)$ such that $f(c)=r$.

In the present case note that $p(x)=2x^5-2x^4+2x^2-2x-1$ is a polynomial, hence continuous, and in particular continuous on the closed interval $[-1/2,0]$. Also,

$$p(0)=-1<0<5/16=p(-1/2).$$

So the theorem applies to $p(x)$ with $a=-1/2$, $b=0$, and $r=0$.

3. The correct matching is:

function	(a)	(b)	(c)	(d)	(e)
derivative	(3)	(4)	(1)	(5)	(2)

4. (a) $\dfrac{d}{dx}\left(\dfrac{x}{1+x}\right) = \dfrac{-x}{(1+x)^2}\dfrac{d}{dx}(1+x) + \dfrac{1}{1+x} = \dfrac{1}{(1+x)^2}$

   (b) $\dfrac{d}{dx}\left(\ln(\sqrt[3]{x})\right) = \dfrac{1}{\sqrt[3]{x}}\dfrac{d}{dx}\left(\sqrt[3]{x}\right) = \dfrac{1}{\sqrt[3]{x}}\dfrac{1}{3}(x)^{-2/3} = \dfrac{1}{3x}$

   We can also begin by rewriting $\ln(\sqrt[3]{x})$ as $(1/3)\ln x$.

   (c) $$\begin{aligned}\frac{d}{dx}\left(x\,\sin^{-1}(1/x)\right) &= \sin^{-1}(1/x) + \frac{x}{\sqrt{1-(1/x)^2}}\frac{d}{dx}(1/x)\\ &= \sin^{-1}(1/x) - \frac{1}{\sqrt{x^2-1}}.\end{aligned}$$

5. We are given the equations:

$$\begin{cases} V &= \frac{1}{3}\pi r^2 h \\ 1 &= h^2 + r^2. \end{cases}$$

We have to maximize V. Eliminating r we get

$$\begin{aligned} V &= \frac{1}{3}(1-h^2)h \\ \frac{dV}{dh} &= \frac{1}{3}\left(1-3h^2\right) = -\left(h-\frac{1}{\sqrt{3}}\right)\left(h+\frac{1}{\sqrt{3}}\right). \end{aligned}$$

Thus the critical points for V are $h = \pm 1/\sqrt{3}$. In terms of the construction of the cone only values of h in $(0,1)$ are meaningful. By inspection of the equation above, $dV/dh > 0$ for $0 < h < 1/\sqrt{3}$, and $dV/dh < 0$ for $1/\sqrt{3} < h < 1$. Thus the maximum value of V is attained when $h = 1/\sqrt{3}$.

When $h = 1/\sqrt{3}$, we have $r^2 = 1 - h^2 = 2/3$. So

$$V_{\mathsf{max}} = \frac{1}{3}\pi(1-h^2)h = \frac{2\pi}{9\sqrt{3}}.$$

6. Differentiating implicitly the equation

$$\sin(\pi xy) = \frac{2}{3}(x+y)$$

we get

$$\pi\cos(\pi xy)\left[y + x\frac{dy}{dx}\right] = \frac{2}{3}\left(1+\frac{dy}{dx}\right) \qquad (1)$$

Setting $x = 1$, $y = 1/2$ we see that $\cos(\pi xy) = 0$, and so from (1), at $(1, 1/2)$, $dy/dx = -1$.

Differentiating (1) implicitly we get

$$-\pi^2 \sin(\pi xy)\left[y + x\frac{dy}{dx}\right]^2 + \pi\cos(\pi xy)\left[2\frac{dy}{dx} + x\frac{d^2y}{dx^2}\right] = \frac{2}{3}\frac{d^2y}{dx^2}\,. \qquad (2)$$

Setting $x = 1$, $y = 1/2$ in (2), we have

$$-\pi^2\left[(1/2) - 1\right]^2 = \frac{2}{3}\frac{d^2y}{dx^2}\,,$$

which says that $d^2y/dx^2 = -(3\pi^2)/8$.

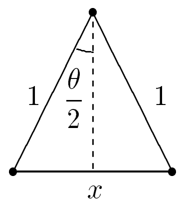
7. Let $f(x) = \tan^{-1} x$. By definition, $f(x)$ is the unique θ in $-\pi/2 < x < \pi/2$ such that $\tan\theta = x$. Hence $f(1) = \pi/4$. Note that

$$f'(1) = \left(\frac{d}{dx}\tan^{-1} x\right)\Bigg|_{x=1} = \left(\frac{1}{1+x^2}\right)\Bigg|_{x=1} = 1/2\,.$$

Taking 1 as base point the linear approximation is

$$f(1.1) \approx f(1) + (.1)f'(1) = \pi/4 + (.1)(.5) = \pi/4 + .05.$$

8. From the diagram on the right we see that



$$x = 2\sin\frac{\theta}{2}\,.$$

We are given that, when $\theta = \pi/3$, then $\dfrac{d\theta}{dt} = 1$.

Differentiating with respect to t and then setting $\theta = \pi/3$ we get

$$\frac{dx}{dt} = \left(\cos\frac{\theta}{2}\right)\frac{d\theta}{dt} = \left(\cos\frac{\pi}{6}\right)(1) = \frac{\sqrt{3}}{2}\,.$$

There are less convenient ways of expressing x in terms of θ. For example the law of cosines gives

$$x^2 = 1^2 + 1^2 - 2(1\cdot 1)\cos\theta = 2(1-\cos\theta).$$

We get the same expression by using the formula for the distance between the points $(\cos\theta, \sin\theta)$ and $(1, 0)$ in the cartesian plane. Rather than taking the square root it is easier to differentiate the equation as it stands:

$$2x\frac{dx}{dt} = 2\sin\theta\frac{d\theta}{dt}\,.$$

Setting $\theta = \pi/3$ gives $x = 1$ and the same result as before since $\sin\pi/3 = \sqrt{3}/2$.

9. (a) In evaluating the limit as $x \to \infty$ of a quotient of polynomials one can ignore all except the highest powers of x in the numerator and denominator. Thus

$$\lim_{x\to\infty} \frac{2x^2-4x+3}{1-x^2} = \lim_{x\to\infty} \frac{2x^2}{-x^2} = -2$$

and similarly $\lim\limits_{x\to-\infty} \dfrac{2x^2-4x+3}{1-x^2} = -2$. Thus $y=-2$ is an asymptote which the graph approaches both as $x\to\infty$ and as $x\to-\infty$.

(b) We have:

$$\begin{aligned}
\lim_{x\to(-1)^-} \frac{2x^2-4x+3}{1-x^2} &= \lim_{x\to(-1)^-} \frac{2x^2-4x+3}{1-x} \lim_{x\to(-1)^-} \frac{1}{1+x} = \frac{9}{2}(-\infty) = -\infty \\
\lim_{x\to(-1)^+} \frac{2x^2-4x+3}{1-x^2} &= \lim_{x\to(-1)^+} \frac{2x^2-4x+3}{1-x} \lim_{x\to(-1)^+} \frac{1}{1+x} = \frac{9}{2}(\infty) = \infty \\
\lim_{x\to1^-} \frac{2x^2-4x+3}{1-x^2} &= \lim_{x\to1^-} \frac{2x^2-4x+3}{1+x} \lim_{x\to1^-} \frac{1}{1-x} = \frac{1}{2}(\infty) = \infty \\
\lim_{x\to1^+} \frac{2x^2-4x+3}{1-x^2} &= \lim_{x\to1^+} \frac{2x^2-4x+3}{1+x} \lim_{x\to1^+} \frac{1}{1-x} = \frac{1}{2}(-\infty) = -\infty.
\end{aligned}$$

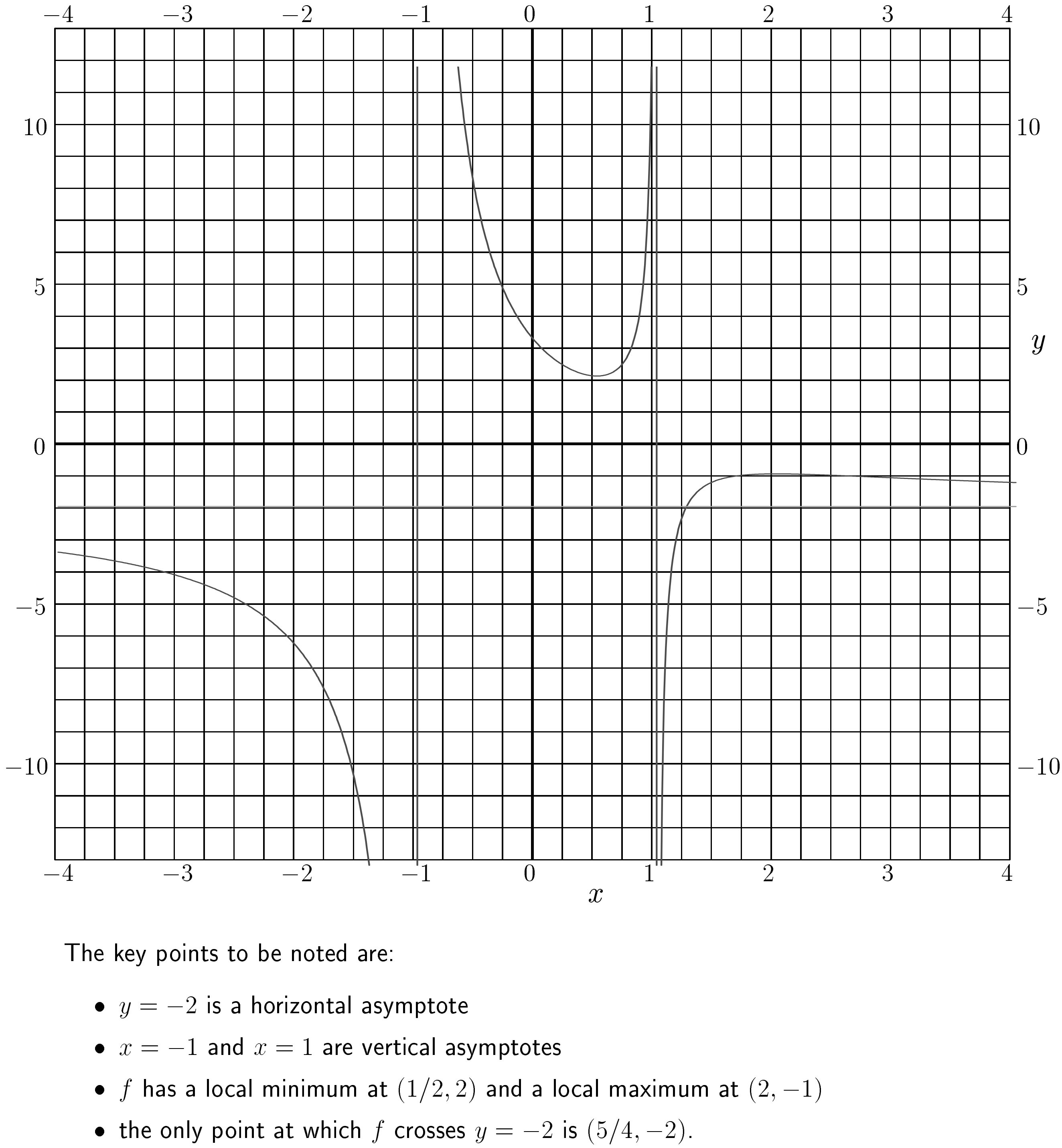
(c) By inspection of the expression for $f'(x)$ we see that $f'(x)>0$ for $x\in(1/2,1)$ and $x\in(1,2)$, and that $f'(x)<0$ for $x\in(-\infty,-1)$, $(-1,1/2)$, and $x\in(2,\infty)$. Hence we have:

$f(x)$ is increasing on $[1/2,1)$ and $(1,2]$

$f(x)$ is decreasing on $(-\infty,-1)$, $(-1,1/2]$, and $[2,\infty)$.

(d) The critical points are $x=1/2$ and $x=2$. Based on the information in part (c) it is clear that f has a local minimum at $x=1/2$ and a local maximum at $x=2$. This is the first derivative test. One could apply the second derivative test if one wished.

(e) Here is the graph:



The key points to be noted are:

- $y = -2$ is a horizontal asymptote
- $x = -1$ and $x = 1$ are vertical asymptotes
- f has a local minimum at $(1/2, 2)$ and a local maximum at $(2, -1)$
- the only point at which f crosses $y = -2$ is $(5/4, -2)$.

10. (a) The speed is diminishing by 20 ft/sec each second. Thus the speed is 0 after 4 seconds.

(b) Integrating once we get the equation

$$\frac{ds}{dt} = -20t + C.$$

Measuring time from the instant when braking begins we have $C = 80$, the initial (uniform) speed. Integrating again gives

$$s = -10t^2 + 80t + C'.$$

Since s and t are measured from the braking point, $C' = 0$. Thus $s = -10t^2 + 80t$ and in the 4 seconds it takes to bring the car to a stop the car travels $-160 + 320 = 160$ ft.

(c) There will be an accident if the car travels more than 40 ft before braking begins. At 80 ft/sec the car travels 40 ft in half a second. So the driver has half a second to react.

11. (a) The differential equation leads to the conclusion that

$$A = Ie^{-kt}$$

where I is the initial amount.

Since the half-life is 5 years, we have $e^{-5k} = 1/2$. Taking natural logarithms,

$$k = \frac{1}{5}\ln 2.$$

(b) For the site to be habitable we need the amount of radioactive material to be reduced to $I/7$. So we need

$$e^{-kt} = 1/7$$

Taking natural logarithms and substitutuing for k, we get $t = (5\ln 7)/\ln 2$ years.

12. (a) $-\pi \leq \theta \leq 3\pi$.

(b) $x = (\theta\cos\theta)/\pi$, $y = (\theta\sin\theta)/\pi$.

(c) It is easy to obtain that $\left.\dfrac{dx}{d\theta}\right|_{\theta=2\pi} = 1$ and $\left.\dfrac{dy}{d\theta}\right|_{\theta=2\pi} = 2\pi$.

So the slope of curve at $\theta = 2\pi$ is $(dy/d\theta)/(dx/d\theta) = 2\pi$.
The equation of the tangent line at $(2, 0)$ is $y = 2\pi(x - 2)$.