Instructions for Calculus Challenge Exam
June 13, 2000, 16:30 – 19:30

1. Do not open the test booklet until told to do so.
2. Check that your name is on the label on the cover page of the test-booklet.
3. Sign your name in the box provided.
4. At the end of the exam only what you have written in the test-booklet will count. You may not submit answers on additional sheets of paper.
5. Write your final answer in the answer box when one is provided.
6. Use the reverse side of the **previous page** if you need more room for your work.
7. NO CALCULATORS or other aids except for a ruler are allowed. A ruler brought to the examination may be ruled ineligible if the invigilator considers it would confer an unfair advantage on its owner.
8. In some questions you are asked to "find" or "compute" values. If the answer comes out to $2/\sqrt{3}$ or $3 + \pi \ln 5$, that is fine. You are not expected to attempt to compute decimal approximations.
9. All angles are assumed to be in radians.
10. Points awarded to parts of questions are shown in square brackets in the left margin.
11. You may write in either pen or pencil, but answers deemed illegible will be ignored.
12. Once the exam begins, please check that you have all 16 pages of the test booklet.
13. The page following contains some facts about trigonometry. Feel free to ignore it or use it as you see fit.

(Use the other side as scratch-paper. Discard at the end of the exam)

Some values of the trigonometric functions

degrees	0	30	45	60	90	120	135	150
radians	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$
$\sin\theta$	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$
$\cos\theta$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0	$-1/2$	$-1/\sqrt{2}$	$-\sqrt{3}/2$

Trigonometric identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\sin(\pi+\theta) = -\sin\theta = \sin(-\theta) \qquad \cos(\pi+\theta) = -\cos\theta = -\cos(-\theta)$$

$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta$$

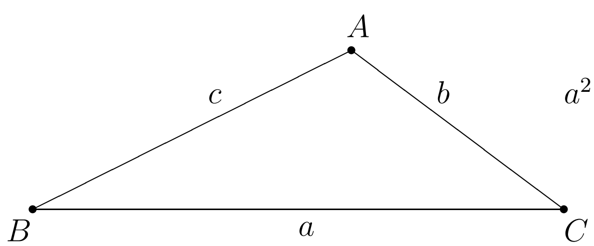
$$\sin(A+B) = \sin A\cos B + \cos A\sin B \qquad \cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right] \qquad \cos A\cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

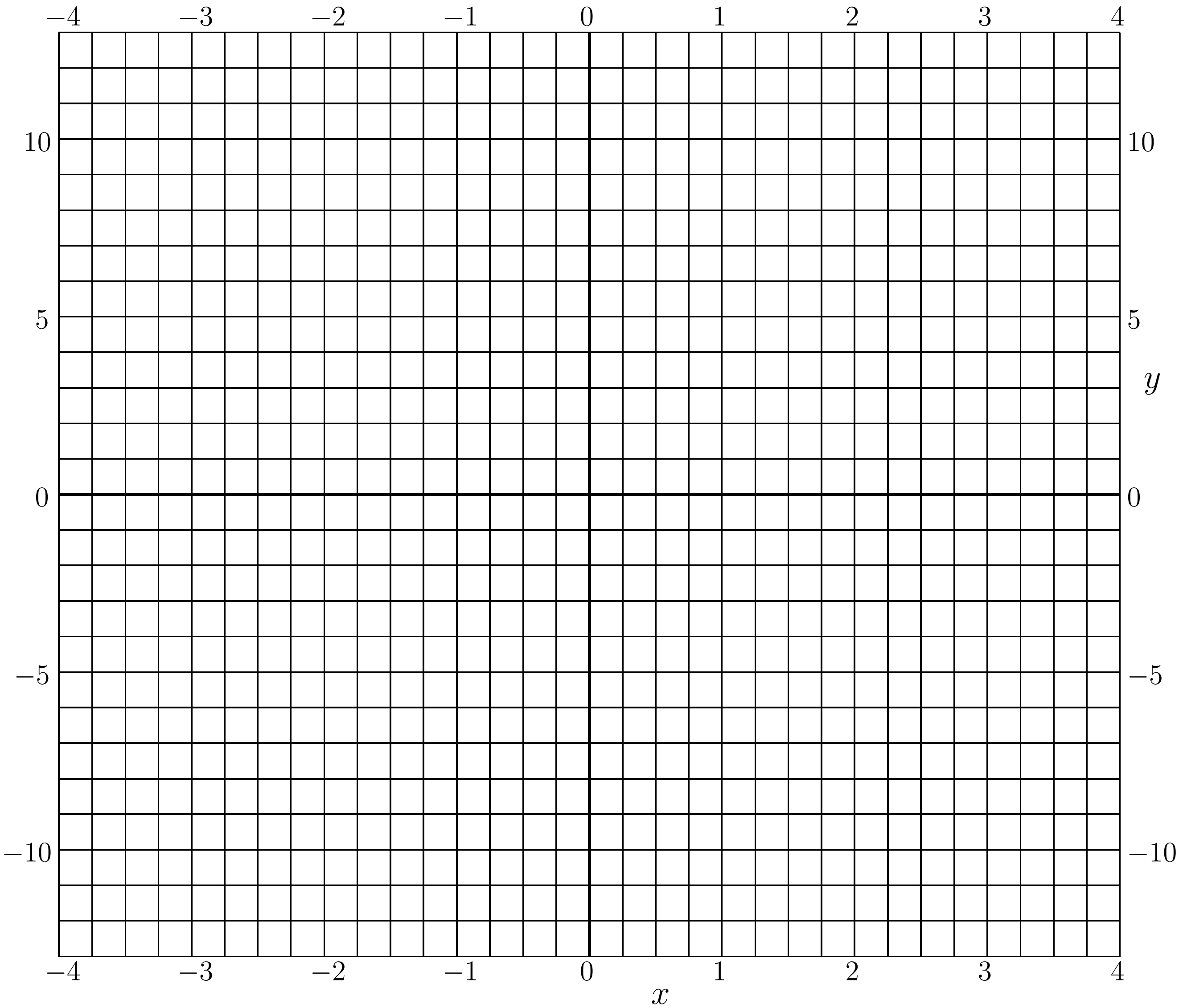
$$\sin 2\theta = 2\sin\theta\cos\theta \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

Law of Cosines



$$a^2 = b^2 + c^2 - 2bc\cos A$$

(Use the other side as scratch-paper. Discard at the end of the exam)



$$f(x) = \frac{2x^2 - 4x + 3}{1 - x^2}, \qquad f'(x) = \frac{(2x-1)(4-2x)}{(1-x^2)^2}\,.$$

(For rough work on the curve-sketching. Discard at the end of the exam)

SFU - UBC - UNBC - UVic

Calculus Challenge Exam

June 13, 2000, 16:30 – 19:30

Host: SIMON FRASER UNIVERSITY

Student signature

INSTRUCTIONS

The instructions are distributed separately. Please read them carefully.

Question	Maximum	Score
1	10	
2	4	
3	6	
4	9	
5	8	
6	7	
7	4	
8	6	
9	20	
10	10	
11	6	
12	10	
Total	100	

1. Compute the following limits.

[3] (a) $\lim_{x \to 1} \dfrac{x^3 - 1}{x^4 - 1}$

ANSWER

SHOW YOUR WORK

[3] (b) $\lim_{x \to 0^+} \dfrac{x + 2}{2 + \sqrt{x}}$

ANSWER

SHOW YOUR WORK

[4] (c) $\lim_{x \to 0} \dfrac{\sqrt{1 - \cos 2x}}{|x|}$

ANSWER

SHOW YOUR WORK

[4] **2.** Define

$$p(x) = 2x^5 - 2x^4 + 2x^2 - 2x - 1.$$

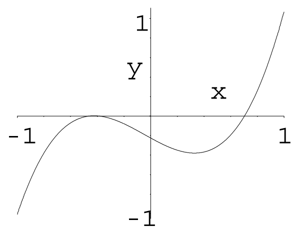
State a general principle (theorem) which tells us that $p(x) = 0$ has a solution in the interval $(-1/2, 0)$.

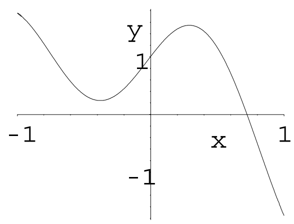
Explain briefly why the principle applies in this case.

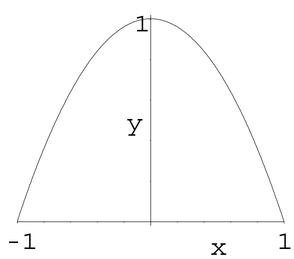
ANSWER

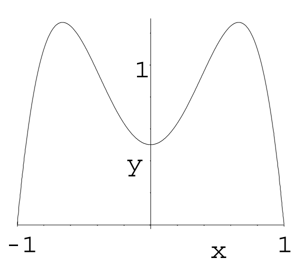
[6] **3.** Match the graph of the function (left column) with the graph of its derivative (right column). Write the number of the derivative below the letter of the corresponding function:

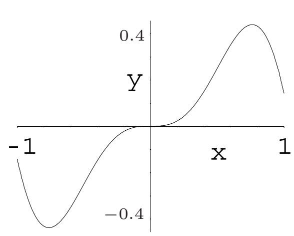
(a)	(b)	(c)	(d)	(e)

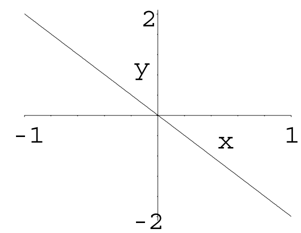
(a)


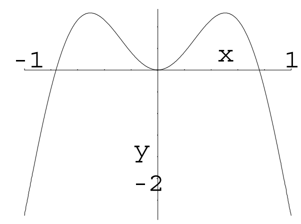
(b)


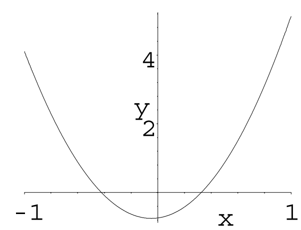
(c)


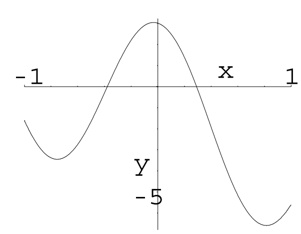
(d)


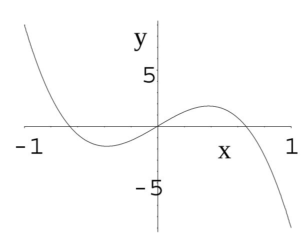
(e)


(1)


(2)


(3)


(4)


(5)


[3] **4.** (a) Find $\dfrac{d}{dx}\left(\dfrac{x}{1+x}\right)$ and simplify your answer.

ANSWER

SHOW YOUR WORK

[3] (b) Find $\dfrac{d}{dx}\left(\ln\left(\sqrt[3]{x}\right)\right)$ and simplify your answer.

ANSWER

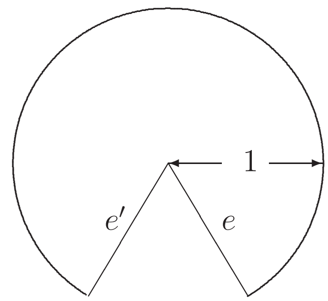
SHOW YOUR WORK

[3] (c) Find $\dfrac{d}{dx}\left(x\sin^{-1}(1/x)\right)$ and simplify your answer.

ANSWER

SHOW YOUR WORK

[8] **5.** A cone is constructed by chopping a segment out of a disc of radius 1 and then gluing the edges e and e' together.



The height h, volume V, and base-radius r of the resulting cone satisfy the equations:

$$V = \frac{1}{3}\pi r^2 h$$
$$1 = h^2 + r^2.$$

Find the maximum possible value of V.

ANSWER

$V_{\mathsf{max}} =$

SHOW YOUR WORK

[7] **6.** Consider the curve defined by the equation

$$\sin(\pi x y) = \frac{2}{3}(x + y)$$

By implicit differentiation compute $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the point $(1, 1/2)$ of the curve.

ANSWER: at $(1, 1/2)$

$\frac{dy}{dx} =$

$\frac{d^2y}{dx^2} =$

SHOW YOUR WORK

[4] **7.** Find an approximate value for $\tan^{-1}(1.1)$ using linear approximation.

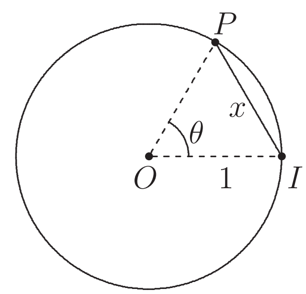
ANSWER

SHOW YOUR WORK

[6] **8.** A point P is moving anticlockwise on a circle of radius 1.

I is the initial position of the point and x denotes the distance from P to I.

It is given that $\dfrac{d\theta}{dt} = 1$ when $\theta = \pi/3$, where t measures time.



Find $\dfrac{dx}{dt}$ when $\theta = \pi/3$.

ANSWER: at $\theta = \pi/3$
$\dfrac{dx}{dt} =$

SHOW YOUR WORK

9. It is given that

$$f(x) = \frac{2x^2 - 4x + 3}{1 - x^2}, \qquad f'(x) = \frac{(2x-1)(4-2x)}{(1-x^2)^2}.$$

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[3] (a) Compute $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

ANSWER

$\lim_{x\to\infty} f(x) =$

$\lim_{x\to-\infty} f(x) =$

SHOW YOUR WORK FOR ONE OF THE LIMITS

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[3] (b) Complete the table of values:

$\displaystyle\lim_{x\to(-1)^-} \frac{2x^2 - 4x + 3}{1 - x^2}$	$-\infty$
$\displaystyle\lim_{x\to(-1)^+} \frac{2x^2 - 4x + 3}{1 - x^2}$	
$\displaystyle\lim_{x\to 1^-} \frac{2x^2 - 4x + 3}{1 - x^2}$	
$\displaystyle\lim_{x\to 1^+} \frac{2x^2 - 4x + 3}{1 - x^2}$	

SHOW YOUR WORK FOR ONE OF THE ENTRIES

[3] (c) List the intervals on which $f(x)$ is increasing and the intervals on which $f(x)$ is decreasing.

ANSWER

$f(x)$ is increasing on:

$f(x)$ is decreasing on:

SHOW YOUR WORK

[3] (d) Classify the critical points of the function $f(x)$ as "local maximum", "local minimum", or "other".

ANSWER

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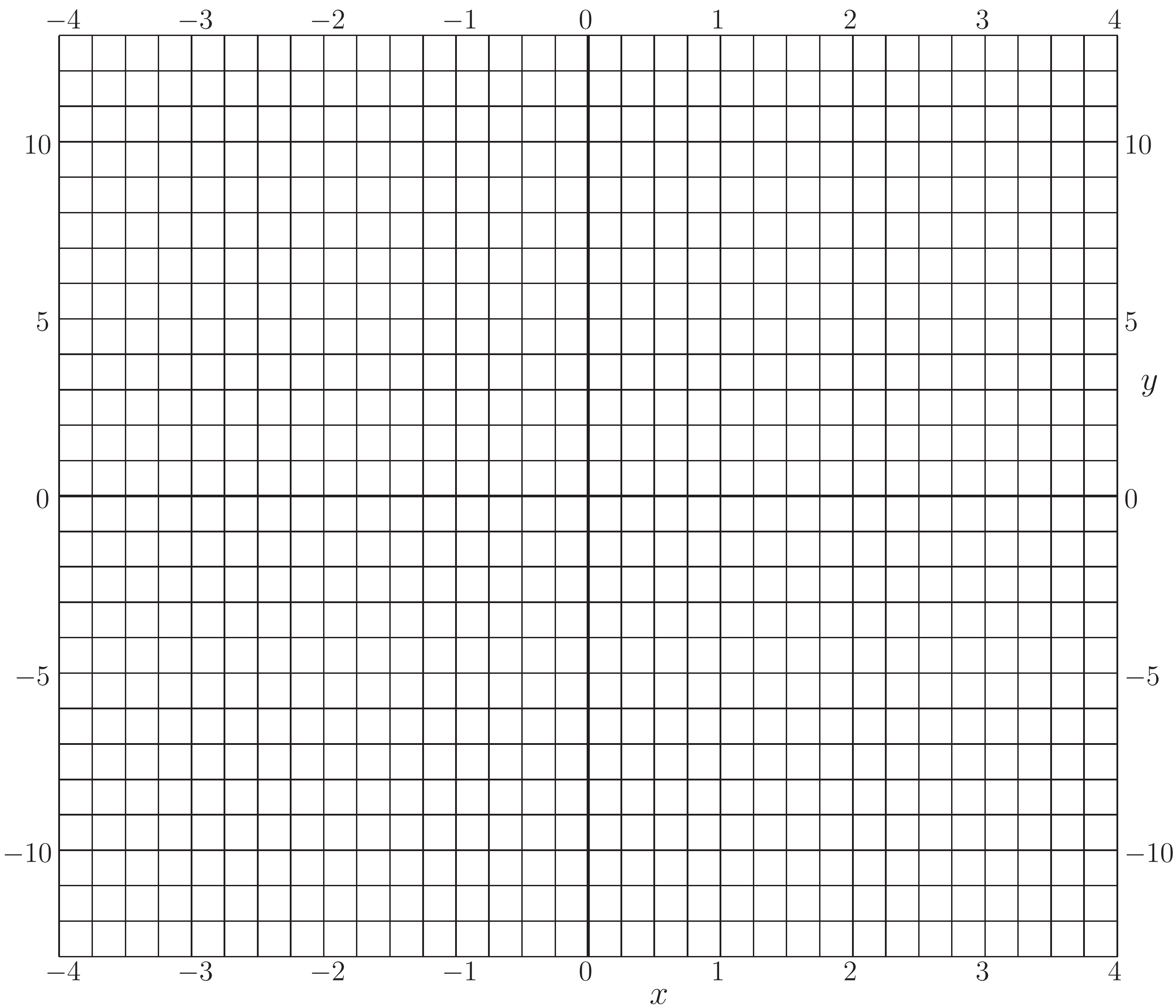
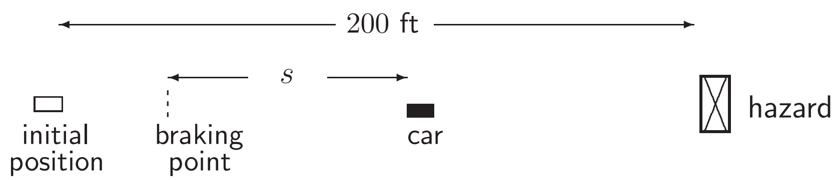
[8] (e) Sketch the part of the graph of $y = f(x)$ which lies in the rectangle $-4 \le x \le 4$, $-13 \le y \le 13$ on the grid provided below.

Mark on the graph all asymptotes, local maxima and minima if any, and points at which the graph crosses its asymptotes if any.





10. A driver is traveling along a straight road at a uniform speed of 80 ft/sec when she suddenly notices a hazard 200 feet ahead. As soon as the brakes are applied the car will decelerate at a rate of 20 ft/sec^2.

Once the brakes are applied the differential equation describing the motion of the car is

$$\frac{d^2s}{dt^2} = -20 \tag{1}$$

where s is the distance (in feet) the car has traveled under braking and t is the time which has elapsed in seconds.

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[2] (a) How many seconds from the moment the brakes are applied does it take for the car to stop?

No work need be shown.

ANSWER

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[6] (b) By integrating the differential equation (1), find the stopping distance, i.e., the distance the car travels from the instant the brakes are applied.

ANSWER

SHOW YOUR WORK

(more space for your answer to **10**(b) *if needed)*

[2] (c) How long does the driver have to react if an accident is to be avoided?

ANSWER

SHOW YOUR WORK

11. The half-life of a certain cobalt isotope is 5 years, i.e., in any sample it takes 5 years for half of the atoms to decay.

After a nuclear accident the concentration of this isotope found at the site is 7 times the maximum level considered acceptable for human habitation.

The law governing radioactive decay says that, if A is the total amount of the isotope at the site t years after the accident, then

$$\frac{dA}{dt} = -kA,$$

where k is a constant > 0.

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[3] (a) Find k.

ANSWER

SHOW YOUR WORK

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[3] (b) Find the number of years it will take for the site to be fit for human occupation.

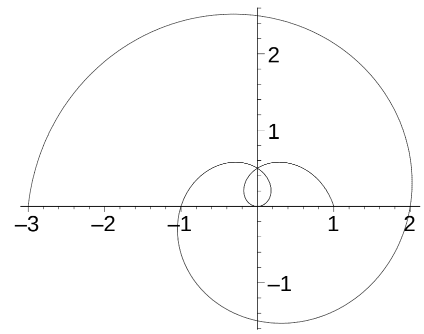
ANSWER

SHOW YOUR WORK

12. On the right of the page is part of the graph of the curve whose equation in polar coordinates is:

$$r = \theta/\pi$$

The axes shown are the usual cartesian axes. According to the usual convention, the polar axis is the same as the positive x-axis.



[3] (a) Write down the interval for θ which corresponds to the portion of the graph actually displayed.

No work need be shown.

ANSWER

[3] (b) Write the curve in parametric form:

$$\begin{cases} x = f(\theta) \\ y = g(\theta) \end{cases}$$

ANSWER

$x =$

$y =$

SHOW YOUR WORK FOR (b)

[4] (c) Find the equation of the tangent line (with respect to cartesian coordinates) at the point $(2, 0)$ which is the point of the curve given by taking $\theta = 2\pi$.

ANSWER

SHOW YOUR WORK FOR (c)