CHAPTER 8

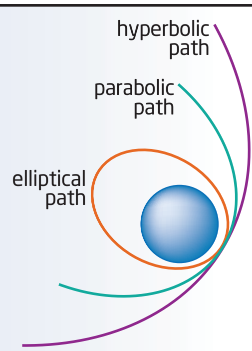
Systems of Equations

What causes that strange feeling in your stomach when you ride a roller coaster? Where do elite athletes get their technical information? How do aerospace engineers determine when and where a rocket will land or what its escape velocity from a planet's surface is? If you start your own business, when can you expect it to make a profit?

The solution to all these questions involves the types of equations that you will work with in this chapter. Systems of equations have applications in science, business, sports, and many other areas, and they are often used as part of a decision-making process.

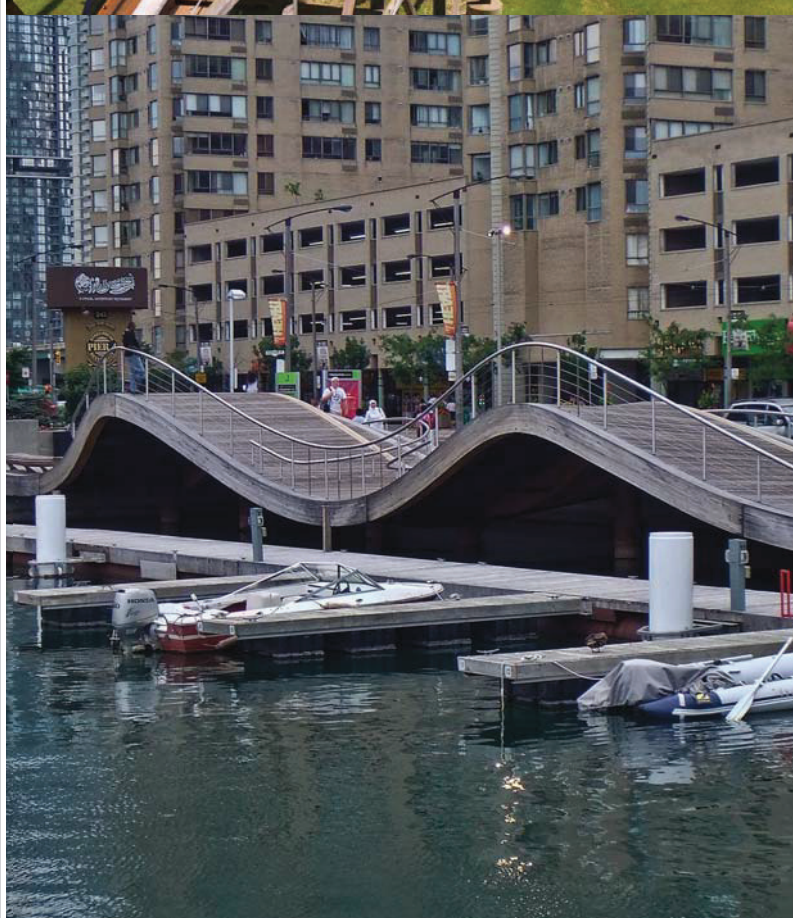
Did You Know?

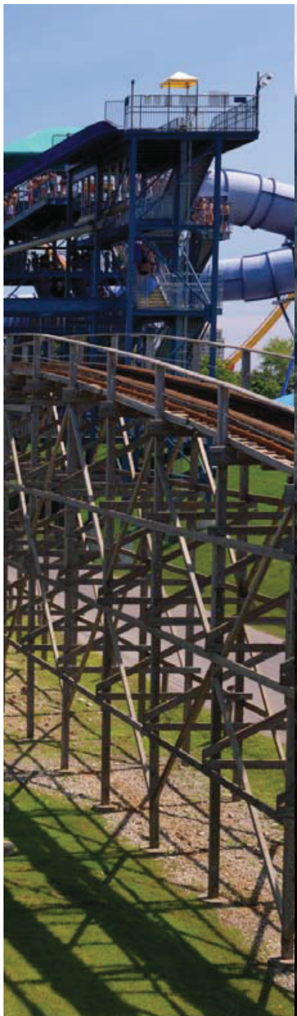
An object can take several different orbital paths. To leave a planet's surface, a rocket must reach escape velocity. The escape velocity is the velocity required to establish a parabolic orbit.

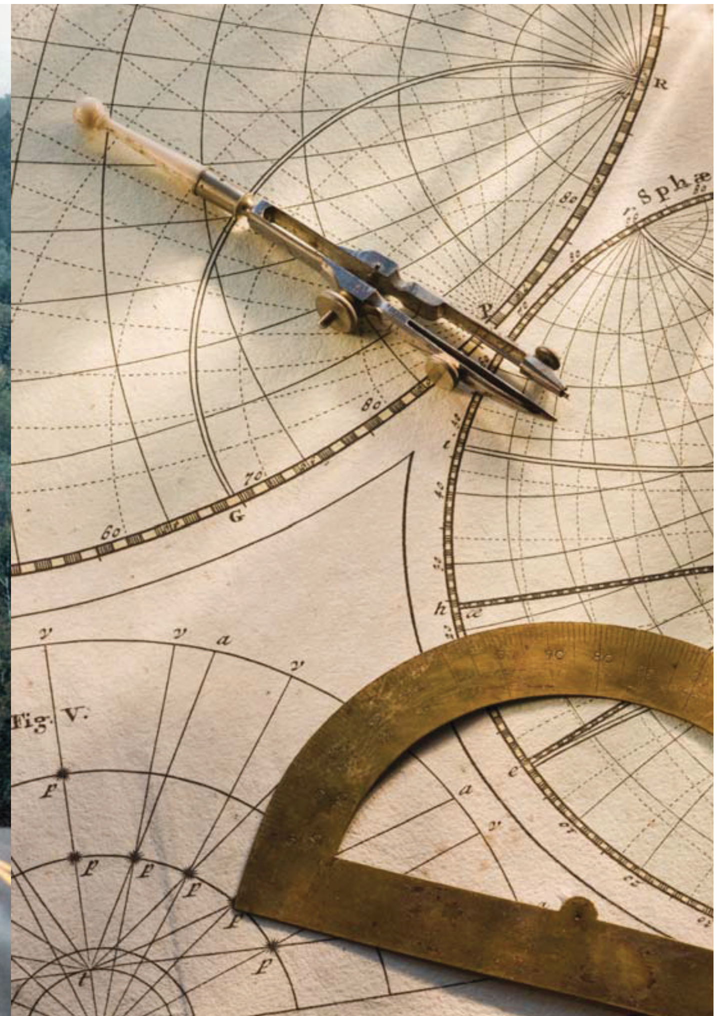


Key Terms

system of linear-quadratic equations
system of quadratic-quadratic equations







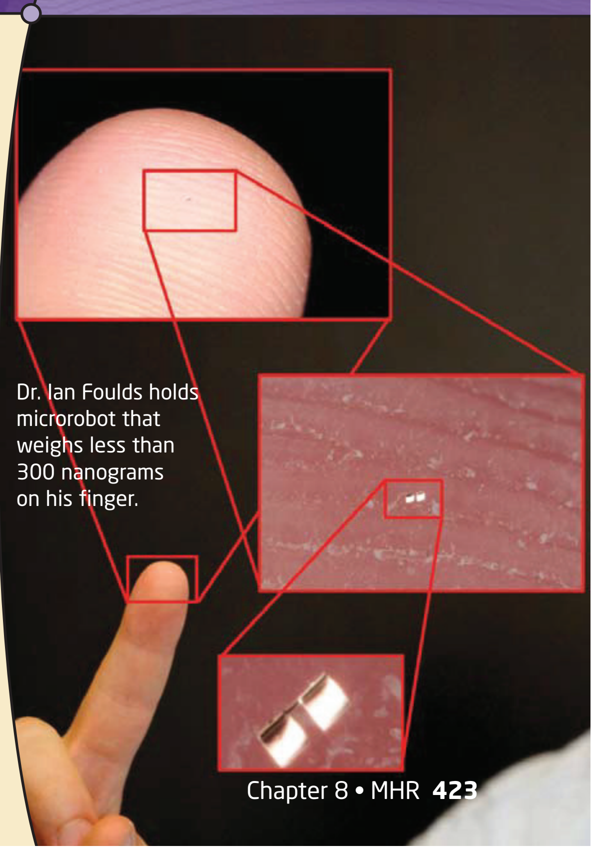
Career Link

The career of a university researcher may include publishing papers, presenting at conferences, and teaching and supervising students doing research in fields that they find interesting. University researchers often also have the opportunity to travel.

Dr. Ian Foulds, from Salmon Arm, British Columbia, works as a university researcher in Saudi Arabia. His research in the field of nanotechnology includes developing surface micromachining processes. Dr. Foulds graduated in electrical engineering, completing his doctorate at Simon Fraser University in Burnaby, British Columbia.

Web Link

To learn more about fields involving research, go to www.mhrprecalc11.ca and follow the links.



Dr. Ian Foulds holds microrobot that weighs less than 300 nanograms on his finger.

8.1

Solving Systems of Equations Graphically

Focus on…

- modelling a situation using a system of linear-quadratic or quadratic-quadratic equations
- determining the solution of a system of linear-quadratic or quadratic-quadratic equations graphically
- interpreting points of intersection and the number of solutions of a system of linear-quadratic or quadratic-quadratic equations
- solving a problem that involves a system of linear-quadratic or quadratic-quadratic equations

Companies that produce items to sell on the open market aim to make a maximum profit. When a company has no, or very few, competitors, it controls the marketplace by deciding the price of the item and the quantity sold. The graph in the Investigate below illustrates the relationship between the various aspects that a company must consider when determining the price and quantity. Notice that the curves intersect at a number of points. What do you know about points of intersection on a graph?

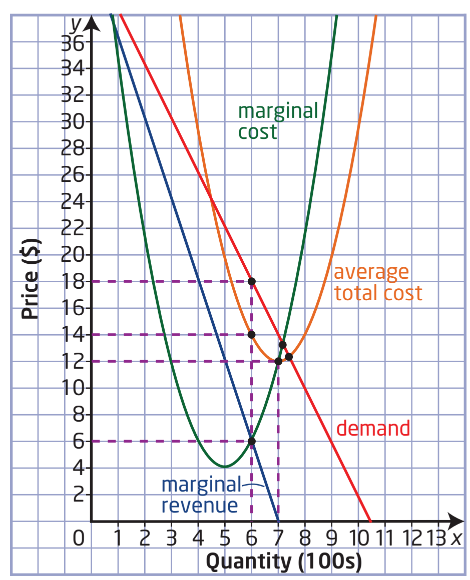
Investigate Solving Systems of Equations Graphically

Work with a partner to discuss your findings.

Part A: Solutions to a System

The graph shows data that a manufacturing company has collected about the business factors for one of its products.

1. The company's profits are maximized when the marginal revenue is equal to the marginal cost. Locate this point on the graph. What is the quantity produced and the price of the item when profits are maximized?



Did You Know?

Economists often work with graphs like the one shown. The marginal cost curve shows the change in total cost as the quantity produced changes, and the marginal revenue curve shows the change in the corresponding total revenue received.

2. When the average total cost is at a minimum, it should equal the marginal cost. Is this true for the graph shown? Explain how you know.

3. A vertical line drawn to represent a production quantity of 600 items intersects all four curves on the graph. Locate the point where this vertical line intersects the demand curve. If the company produces more than 600 items, will the demand for their product increase or decrease? Explain.

Part B: Number of Possible Solutions

4. a) The manufacturing company's graph shows three examples of systems involving a parabola and a line. Identify two business factors that define one of these systems.

   b) Consider other possible systems involving one line and one parabola. Make a series of sketches to illustrate the different ways that a line and a parabola can intersect. In other words, explore the possible numbers of solutions to a **system of linear-quadratic equations** graphically.

5. a) The manufacturing company's graph shows an example of a system involving two parabolas. Identify the business factors that define this system.

   b) Consider other possible systems involving two parabolas. Make a series of sketches to illustrate the different ways that two parabolas can intersect. In other words, explore the possible numbers of solutions to a **system of quadratic-quadratic equations**.

Reflect and Respond

6. Explain how you could determine the solution(s) to a system of linear-quadratic or quadratic-quadratic equations graphically.

7. Consider the coordinates of the point of intersection of the marginal revenue curve and the marginal cost curve, (600, 6). How are the coordinates related to the equations for marginal revenue and marginal cost?

system of linear-quadratic equations

- a linear equation and a quadratic equation involving the same variables
- a graph of the system involves a line and a parabola

system of quadratic-quadratic equations

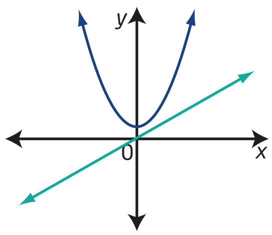
- two quadratic equations involving the same variables
- the graph involves two parabolas

Link the Ideas

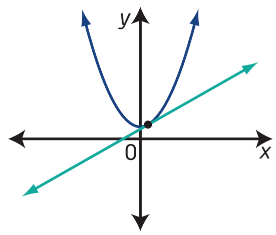
Any ordered pair (x, y) that satisfies both equations in a system of linear-quadratic or quadratic-quadratic equations is a solution of the system.

For example, the point (2, 4) is a solution of the system
$y = x + 2$
$y = x^2$
The coordinates $x = 2$ and $y = 4$ satisfy both equations.

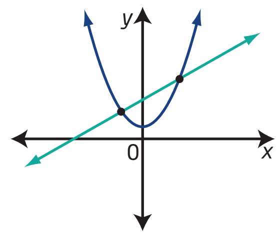
A system of linear-quadratic or quadratic-quadratic equations may have no real solution, one real solution, or two real solutions. A quadratic-quadratic system of equations may also have an infinite number of real solutions.



No point of
intersection
No solution



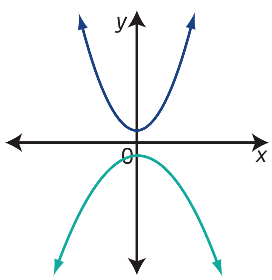
One point of
intersection
One solution



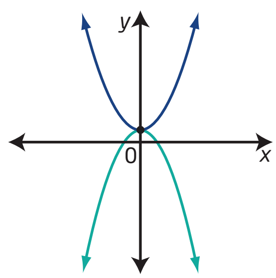
Two points of
intersection
Two solutions

Can two parabolas that both open downward have no points of intersection? one point? two points? Explain how.

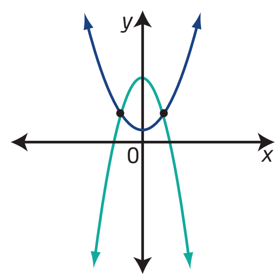
What would the graph of a system of quadratic-quadratic equations with an infinite number of solutions look like?



No point of
intersection
No solution



One point of
intersection
One solution



Two points of
intersection
Two solutions

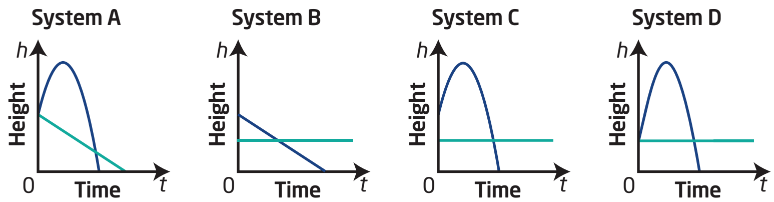
Example 1

Relate a System of Equations to a Context

Blythe Hartley, of Edmonton, Alberta, is one of Canada's best springboard divers. She is doing training dives from a 3-m springboard. Her coach uses video analysis to plot her height above the water.

a) Which system could represent the scenario? Explain your choice and why the other graphs do not model this situation.

b) Interpret the point(s) of intersection in the system you chose.



Solution

a) System D, a linear-quadratic system, represents the scenario. The board height is fixed and the diver's parabolic path makes sense relative to this height. She starts on the board, jumps to her maximum height, and then her height decreases as she heads for the water.

The springboard is fixed at a height of 3 m above the water. Its height must be modelled by a constant linear function, so eliminate System A. The path of the dive is parabolic, with the height of the diver modelled by a quadratic function, so eliminate System B. Blythe starts her dive from the 3-m board, so eliminate System C.

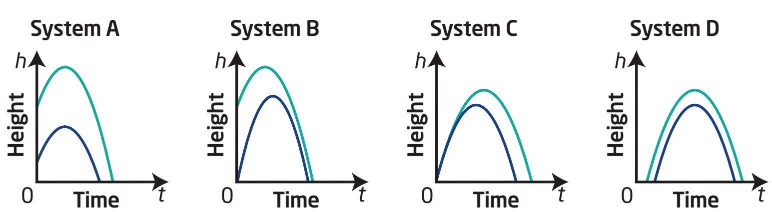
b) The points of intersection in System D represent the two times when Blythe's height above the water is the same as the height of the diving board.

Your Turn

Two divers start their dives at the same time. One diver jumps from a 1-m springboard and the other jumps from a 3-m springboard. Their heights above the water are plotted over time.

a) Which system could model this scenario? Explain your choice. Tell why the other graphs could not model this situation.

b) Explain why there is no point of intersection in the graph you chose.



Example 2

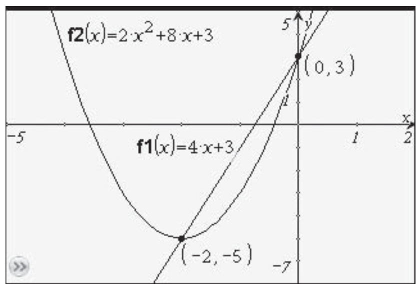
Solve a System of Linear-Quadratic Equations Graphically

a) Solve the following system of equations graphically:
$4x - y + 3 = 0$
$2x^2 + 8x - y + 3 = 0$

b) Verify your solution.

Solution

a) Graph the corresponding functions. Adjust the dimensions of the graph so that the points of intersection are visible. Then, use the intersection feature.



If you are using paper and pencil, it may be more convenient to write the linear equation in slope-intercept form and the quadratic equation in vertex form.

From the graph, the points of intersection are (0, 3) and (−2, −5).

b) Verify the solutions by substituting into the original equations.

Verify the solution (0, 3):
Substitute $x = 0$, $y = 3$ into the original equations.

Left Side	Right Side
$4x - y + 3$	0
$= 4(0) - (3) + 3$	
$= 0$	

Left Side = Right Side

Left Side	Right Side
$2x^2 + 8x - y + 3$	0
$= 2(0)^2 + 8(0) - (3) + 3$	
$= 0$	

Left Side = Right Side

Verify the solution $(-2, -5)$:
Substitute $x = -2$, $y = -5$ into the original equations.

Left Side	Right Side
$4x - y + 3$	0
$= 4(-2) - (-5) + 3$	
$= -8 + 5 + 3$	
$= 0$	

Left Side = Right Side

Left Side	Right Side
$2x^2 + 8x - y + 3$	0
$= 2(-2)^2 + 8(-2) - (-5) + 3$	
$= 8 - 16 + 5 + 3$	
$= 0$	

Left Side = Right Side

Both solutions are correct.

The solutions to the system are $(-2, -5)$ and $(0, 3)$.

Your Turn

Solve the system graphically and verify your solution.

$x - y + 1 = 0$

$x^2 - 6x + y + 3 = 0$

Example 3

Solve a System of Quadratic-Quadratic Equations Graphically

a) Solve:

$2x^2 - 16x - y = -35$

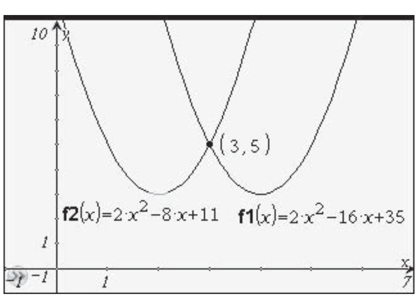
$2x^2 - 8x - y = -11$

How many solutions do you think are possible in this situation?

b) Verify your solution.

Solution

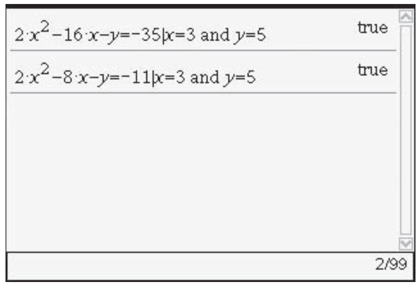
a) Graph the corresponding functions for both equations on the same coordinate grid.



From the graph, the point of intersection is (3, 5).

How do you know that the graphs do not intersect again at a greater value of *y*?

b) Method 1: Use Technology



Method 2: Use Paper and Pencil

Left Side	Right Side	Left Side	Right Side
$2x^2 - 16x - y$	-35	$2x^2 - 8x - y$	-11
$= 2(3)^2 - 16(3) - 5$		$= 2(3)^2 - 8(3) - 5$	
$= 18 - 48 - 5$		$= 18 - 24 - 5$	
$= -35$		$= -11$	
Left Side = Right Side		Left Side = Right Side	

Since the ordered pair (3, 5) satisfies both equations, it is the solution to the system.

Your Turn

Solve the system graphically and verify your solution.

$2x^2 + 16x + y = -26$

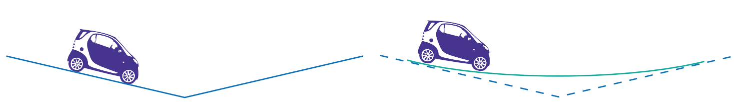
$x^2 + 8x - y = -19$

How many solutions do you think are possible in this situation?

Example 4

Apply a System of Linear-Quadratic Equations

Engineers use vertical curves to improve the comfort and safety of roadways. Vertical curves are parabolic in shape and are used for transitions from one straight grade to another. Each grade line is tangent to the curve.



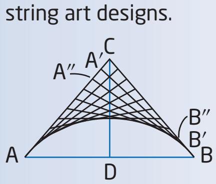
What does it mean for each grade line to be tangent to the curve?

There are several vertical curves on the Trans-Canada Highway through the Rocky Mountains. To construct a vertical curve, surveyors lay out a grid system and mark the location for the beginning of the curve and the end of the curve.

Suppose surveyors model the first grade line for a section of road with the linear equation $y = -0.06x + 2.6$, the second grade line with the linear equation $y = 0.09x + 2.35$, and the parabolic curve with the quadratic equation $y = 0.0045x^2 + 2.8$.

Did You Know?

You can use tangent lines to draw parabolas. Draw a horizontal line segment AB. At the midpoint of AB, draw a height CD. Draw lines CA and CB (these are the first tangent lines to the parabola). Mark the same number of equally spaced points on CA and CB. Connect the point A′ on CA (next to C) to the point B′ on CB (next to B). Then connect A″ (next to A′) to B″ (next to B′), and so on. Follow this pattern for successive pairs of points until all points on CB have been connected to the corresponding points on CA. This technique is the basis of most string art designs.



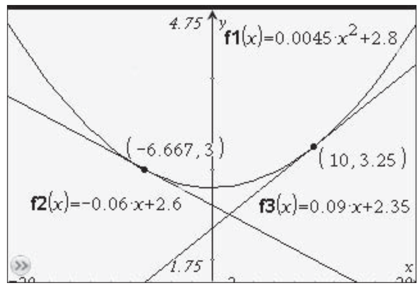
a) Write the two systems of equations that would be used to determine the coordinates of the points of tangency.
b) Using graphing technology, show the surveyor's layout of the vertical curve.
c) Determine the coordinates of the points of tangency graphically, to the nearest hundredth.
d) Interpret each point of tangency.

Solution

a) The points of tangency are where the lines touch the parabola. The two systems of equations to use are

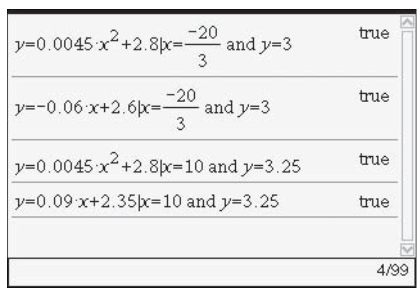
$y = -0.06x + 2.6$ and $y = 0.09x + 2.35$
$y = 0.0045x^2 + 2.8$ $\quad$ $y = 0.0045x^2 + 2.8$

b) Graph all three equations. You may need to adjust the window to see the points of tangency.



c) Use the intersection feature to determine the coordinates of the two points of tangency.

Verify using the calculator.



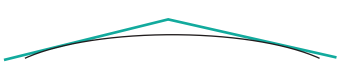
To the nearest hundredth, the points of tangency are (−6.67, 3.00) and (10.00, 3.25).

Could this solution be found using pencil and paper? Explain.

d) This means that the vertical curve starts at the location (−6.67, 3.00) on the surveyor's grid system and ends at the location (10.00, 3.25).

Your Turn

Another section of road requires the curve shown in the diagram. The grade lines are modelled by the equations $y = 0.08x + 6.2$ and $y = -0.075x + 6.103\,125$. The curve is modelled by the equation $y = -0.002x^2 + 5.4$.



a) Write the two systems of equations to use to determine the coordinates of the beginning and the end of the vertical curve on a surveyor's grid.
b) Using graphing technology, show the surveyor's layout of the vertical curve.
c) Determine the coordinates of each end of this vertical curve, to the nearest hundredth.

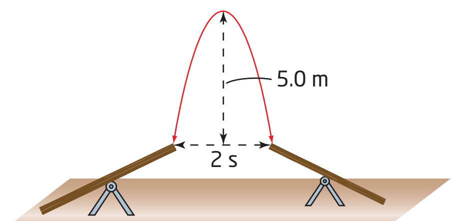
Example 5

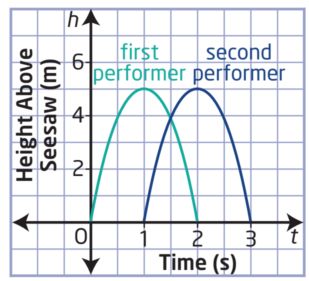
Model a Situation Using a System of Equations

Suppose that in one stunt, two Cirque du Soleil performers are launched toward each other from two slightly offset seesaws. The first performer is launched, and 1 s later the second performer is launched in the opposite direction. They both perform a flip and give each other a high five in the air. Each performer is in the air for 2 s. The height above the seesaw versus time for each performer during the stunt is approximated by a parabola as shown. Their paths are shown on a coordinate grid.

Did You Know?

Cirque du Soleil is a Québec-based entertainment company that started in 1984 with 20 street performers. The company now has over 4000 employees, including 1000 performers, and performs worldwide. Their dramatic shows combine circus arts with street entertainment.





a) Determine the system of equations that models the performers' height during the stunt.
b) Solve the system graphically using technology.
c) Interpret your solution with respect to this situation.

Solution

a) For the first performer (teal parabola), the vertex of the parabola is at (1, 5).

Use the vertex form for a parabola: $h = a(t - p)^2 + q$

Substitute the coordinates of the vertex:
$h = a(t - 1)^2 + 5$

The point (0, 0) is on the parabola. Substitute and solve for a:
$0 = a(0 - 1)^2 + 5$
$-5 = a$

The equation for the height of the first performer versus time is $h = -5(t - 1)^2 + 5$.

For the second performer (blue parabola), the vertex of the parabola is at (2, 5).

Use the vertex form for a parabola: $h = a(t - p)^2 + q$

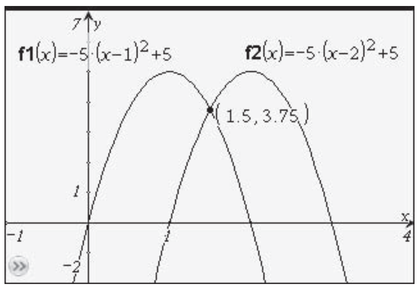
Then, the equation with the vertex values substituted is
$h = a(t - 2)^2 + 5$

The point (1, 0) is on the parabola. Substitute and solve for a:
$0 = a(1 - 2)^2 + 5$
$-5 = a$

The equation for the height of the second performer versus time is $h = -5(t - 2)^2 + 5$.

The system of equations that models the performers' heights is
$h = -5(t - 1)^2 + 5$
$h = -5(t - 2)^2 + 5$

b) Use a graphing calculator to graph the system. Use the intersection feature to find the point of intersection.



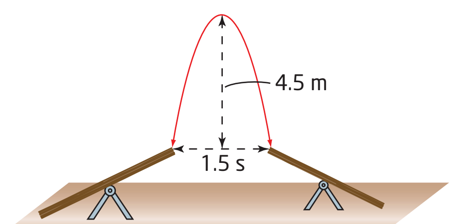
How can you verify this solution?

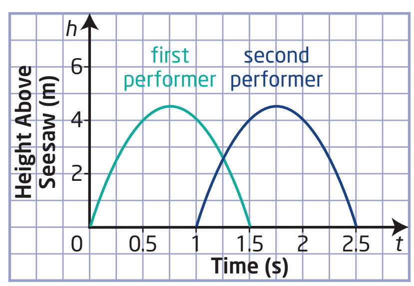
The system has one solution: (1.5, 3.75).

c) The solution means that the performers are at the same height, 3.75 m above the seesaw, at the same time, 1.5 s after the first performer is launched into the air. This is 0.5 s after the second performer starts the stunt. This is where they give each other a high five.

Your Turn

At another performance, the heights above the seesaw versus time for the performers during the stunt are approximated by the parabola shown. Assume again that the second performer starts 1 s after the first performer. Their paths are shown on a coordinate grid.

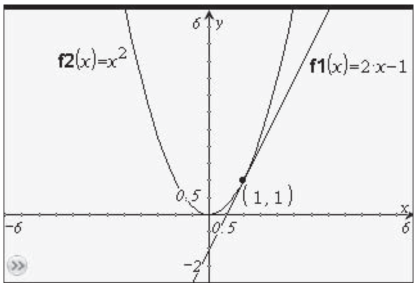


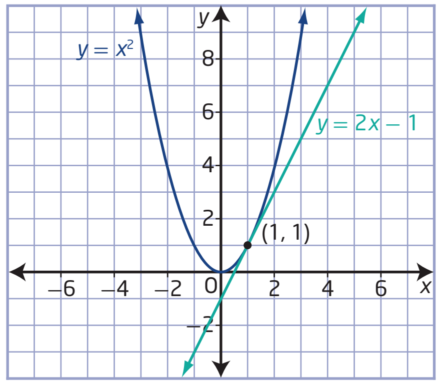


a) Determine the system of equations that models the performers' height during the stunt.
b) Solve the system graphically using technology.
c) Interpret your solution with respect to this situation.

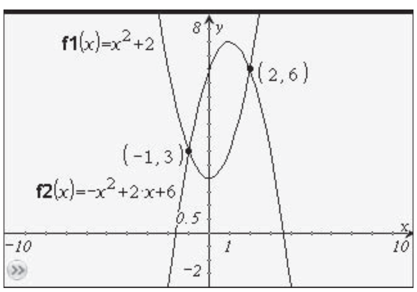
Key Ideas

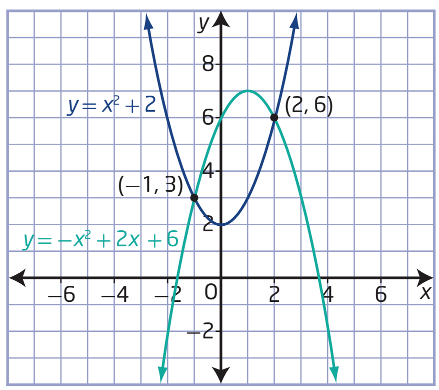
- Any ordered pair (x, y) that satisfies both equations in a linear-quadratic system or in a quadratic-quadratic system is a solution to the system.
- The solution to a system can be found graphically by graphing both equations on the same coordinate grid and finding the point(s) of intersection.





Since there is only one point of intersection, the linear-quadratic system shown has one solution, (1, 1).





Since there are two points of intersection, the quadratic-quadratic system shown has two solutions, approximately (−1, 3) and (2, 6).

- Systems of linear-quadratic equations may have no real solution, one real solution, or two real solutions.
- Systems of quadratic-quadratic equations may have no real solution, one real solution, two real solutions, or an infinite number of real solutions.

Check Your Understanding

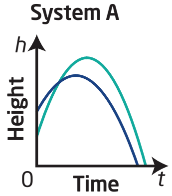
Practise

Where necessary, round answers to the nearest hundredth.

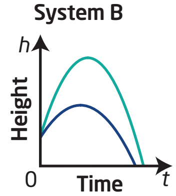
1. The Canadian Arenacross Championship for motocross was held in Penticton, British Columbia, in March 2010. In the competition, riders launch their bikes off jumps and perform stunts. The height above ground level of one rider going off two different jumps at the same speed is plotted. Time is measured from the moment the rider leaves the jump. The launch height and the launch angle of each jump are different.

a) Which system models the situation? Explain your choice. Explain why the other graphs do not model this situation.

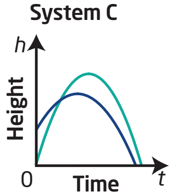
System A



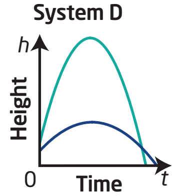
System B



System C



System D



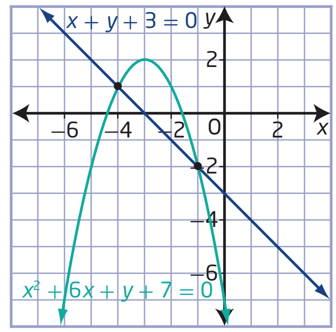
b) Interpret the point(s) of intersection for the graph you selected.

2. Verify that (0, −5) and (3, −2) are solutions to the following system of equations.

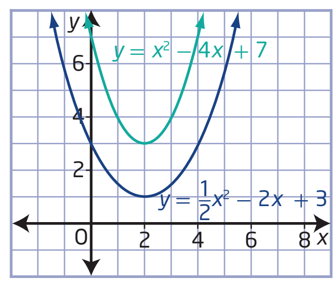
$y = -x^2 + 4x - 5$
$y = x - 5$

3. What type of system of equations is represented in each graph? Give the solution(s) to the system.

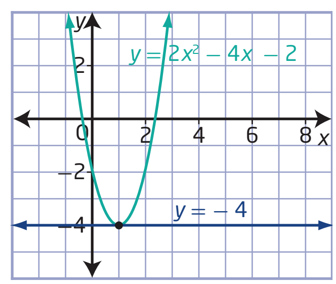
a)



b)



c)



4. Solve each system by graphing. Verify your solutions.

a) $y = x + 7$
$y = (x + 2)^2 + 3$

b) $f(x) = -x + 5$
$g(x) = \frac{1}{2}(x - 4)^2 + 1$

c) $x^2 + 16x + y = -59$
$x - 2y = 60$

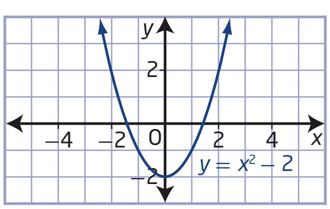
d) $x^2 + y - 3 = 0$
$x^2 - y + 1 = 0$

e) $y = x^2 - 10x + 32$
$y = 2x^2 - 32x + 137$

5. Solve each system by graphing. Verify your solutions.
   a) $h = d^2 - 16d + 60$
   $h = 12d - 55$
   b) $p = 3q^2 - 12q + 17$
   $p = -0.25q^2 + 0.5q + 1.75$
   c) $2v^2 + 20v + t = -40$
   $5v + 2t + 26 = 0$
   d) $n^2 + 2n - 2m - 7 = 0$
   $3n^2 + 12n - m + 6 = 0$
   e) $0 = t^2 + 40t - h + 400$
   $t^2 = h + 30t - 225$

Apply

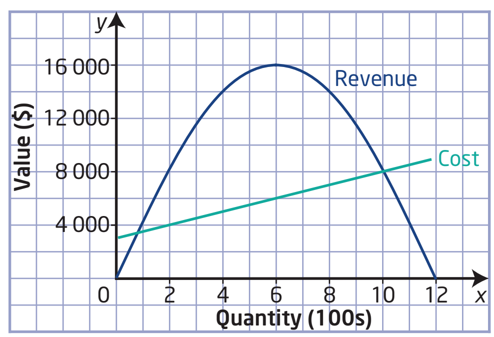
6. Sketch the graph of a system of two quadratic equations with only one real solution. Describe the necessary conditions for this situation to occur.
7. For each situation, sketch a graph to represent a system of quadratic-quadratic equations with two real solutions, so that the two parabolas have
   a) the same axis of symmetry
   b) the same axis of symmetry and the same y-intercept
   c) different axes of symmetry but the same y-intercept
   d) the same x-intercepts
8. Given the graph of a quadratic function as shown, determine the equation of a line such that the quadratic function and the line form a system that has
   a) no real solution
   b) one real solution
   c) two real solutions



9. Every summer, the Folk on the Rocks Music Festival is held at Long Lake in Yellowknife, Northwest Territories.

Dene singer/songwriter, Leela Gilday from Yellowknife.

Jonas has been selling shirts in the Art on the Rocks area at the festival for the past 25 years. His total costs (production of the shirts plus 15% of all sales to the festival) and the revenue he receives from sales (he has a variable pricing scheme) are shown on the graph below.



   a) What are the solutions to this system? Give answers to the nearest hundred.
   b) Interpret the solution and its importance to Jonas.
   c) You can determine the profit using the equation Profit = Revenue − Cost. Use the graph to estimate the quantity that gives the greatest profit. Explain why this is the quantity that gives him the most profit.

10. Vertical curves are used in the construction of roller coasters. One downward-sloping grade line, modelled by the equation $y = -0.04x + 3.9$, is followed by an upward-sloping grade line modelled by the equation $y = 0.03x + 2.675$. The vertical curve between the two lines is modelled by the equation $y = 0.001x^2 - 0.04x + 3.9$. Determine the coordinates of the beginning and the end of the curve.

11. A car manufacturer does performance tests on its cars. During one test, a car starts from rest and accelerates at a constant rate for 20 s. Another car starts from rest 3 s later and accelerates at a faster constant rate. The equation that models the distance the first car travels is $d = 1.16t^2$, and the equation that models the distance the second car travels is $d = 1.74(t - 3)^2$, where t is the time, in seconds, after the first car starts the test, and d is the distance, in metres.

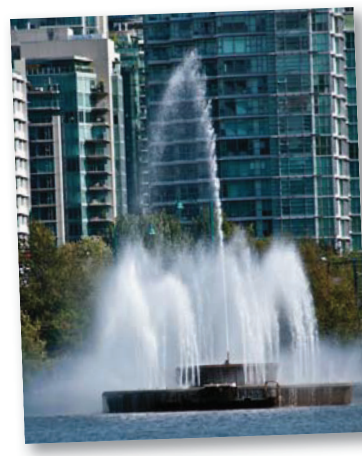
a) Write a system of equations that could be used to compare the distance travelled by both cars.

b) In the context, what is a suitable domain for the graph? Sketch the graph of the system of equations.

c) Graphically determine the approximate solution to the system.

d) Describe the meaning of the solution in the context.

12. Jubilee Fountain in Lost Lagoon is a popular landmark in Vancouver's Stanley Park. The streams of water shooting out of the fountain follow parabolic paths. Suppose the tallest stream in the middle is modelled by the equation $h = -0.3125d^2 + 5d$, one of the smaller streams is modelled by the equation $h = -0.85d^2 + 5.11d$, and a second smaller stream is modelled by the equation $h = -0.47d^2 + 3.2d$, where h is the height, in metres, of the water stream and d is the distance, in metres, from the central water spout.



a) Solve the system $h = -0.3125d^2 + 5d$ and $h = -0.85d^2 + 5.11d$ graphically. Interpret the solution.

b) Solve the system of equations involving the two smaller streams of water graphically. Interpret the solution.

Did You Know?

Jubilee Fountain was built in 1936 to commemorate the city of Vancouver's golden jubilee (50th birthday).

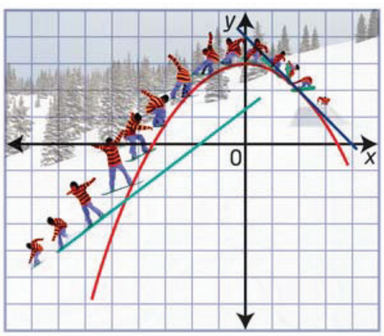
13. The sum of two integers is 21. Fifteen less than double the square of the smaller integer gives the larger integer.

a) Model this information with a system of equations.

b) Solve the system graphically. Interpret the solution.

c) Verify your solution.

14. A Cartesian plane is superimposed over a photograph of a snowboarder completing a 540° Front Indy off a jump. The blue line is the path of the jump and can be modelled by the equation $y = -x + 4$. The red parabola is the path of the snowboarder during the jump and can be modelled by the equation $y = -\frac{1}{4}x^2 + 3$. The green line is the mountainside where the snowboarder lands and can be modelled by the equation $y = \frac{3}{4}x + \frac{5}{4}$.



a) Determine the solutions to the linear-quadratic systems: the blue line and the parabola, and the green line and the parabola.

b) Explain the meaning of the solutions in this context.

15. A frog jumps to catch a grasshopper. The frog reaches a maximum height of 25 cm and travels a horizontal distance of 100 cm. A grasshopper, located 30 cm in front of the frog, starts to jump at the same time as the frog. The grasshopper reaches a maximum height of 36 cm and travels a horizontal distance of 48 cm. The frog and the grasshopper both jump in the same direction.

a) Consider the frog's starting position to be at the origin of a coordinate grid. Draw a diagram to model the given information.

b) Determine a quadratic equation to model the frog's height compared to the horizontal distance it travelled and a quadratic equation to model the grasshopper's height compared to the horizontal distance it travelled.

c) Solve the system of two equations.

d) Interpret your solution in the context of this problem.

Extend

16. The Greek mathematician, Menaechmus (about 380 B.C.E. to 320 B.C.E.) was one of the first to write about parabolas. His goal was to solve the problem of "doubling the cube." He used the intersection of curves to find x and y so that $\frac{a}{x} = \frac{x}{y} = \frac{y}{2a}$, where a is the side of a given cube and x is the side of a cube that has twice the volume. Doubling a cube whose side length is 1 cm is equivalent to solving $\frac{1}{x} = \frac{x}{y} = \frac{y}{2}$.

a) Use a system of equations to graphically solve $\frac{1}{x} = \frac{x}{y} = \frac{y}{2}$.

b) What is the approximate side length of a cube whose volume is twice the volume of a cube with a side length of 1 cm?

c) Verify your answer.

d) Explain how you could use the volume formula to find the side length of a cube whose volume is twice the volume of a cube with a side length of 1 cm. Why was Menaechmus unable to use this method to solve the problem?

Did You Know?

Duplicating the cube is a classic problem of Greek mathematics: *Given the length of an edge of a cube, construct a second cube that has double the volume of the first.* The problem was to find a ruler-and-compasses construction for the cube root of 2. Legend has it that the oracle at Delos requested that this construction be performed to appease the gods and stop a plague. This problem is sometimes referred to as the Delian problem.

17. The solution to a system of two equations is (−1, 2) and (2, 5).

a) Write a linear-quadratic system of equations with these two points as its solutions.

b) Write a quadratic-quadratic system of equations with these two points as its only solutions.

c) Write a quadratic-quadratic system of equations with these two points as two of infinitely many solutions.

18. Determine the possible number of solutions to a system involving two quadratic functions and one linear function. Support your work with a series of sketches. Compare your work with that of a classmate.

Create Connections

19. Explain the similarities and differences between linear systems and the systems you studied in this section.

20. Without graphing, use your knowledge of linear and quadratic equations to determine whether each system has no solution, one solution, two solutions, or an infinite number of solutions. Explain how you know.

a) $y = x^2$
$y = x + 1$

b) $y = 2x^2 + 3$
$y = -2x - 5$

c) $y = (x - 4)^2 + 1$
$y = \frac{1}{3}(x - 4)^2 + 2$

d) $y = 2(x + 8)^2 - 9$
$y = -2(x + 8)^2 - 9$

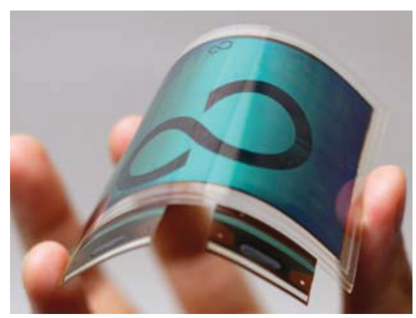
e) $y = 2(x - 3)^2 + 1$
$y = -2(x - 3)^2 - 1$

f) $y = (x + 5)^2 - 1$
$y = x^2 + 10x + 24$

Project Corner Nanotechnology

Nanotechnology has applications in a wide variety of areas.

- **Electronics:** Nanoelectronics will produce new devices that are small, require very little electricity, and produce little (if any) heat.
- **Energy:** There will be advances in solar power, hydrogen fuel cells, thermoelectricity, and insulating materials.
- **Health Care:** New drug-delivery techniques will be developed that will allow medicine to be targeted directly at a disease instead of the entire body.
- **The Environment:** Renewable energy, more efficient use of resources, new filtration systems, water purification processes, and materials that detect and clean up environmental contaminants are some of the potential eco-friendly uses.
- **Everyday Life:** Almost all areas of your life may be improved by nanotechnology: from the construction of your house, to the car you drive, to the clothes you wear.



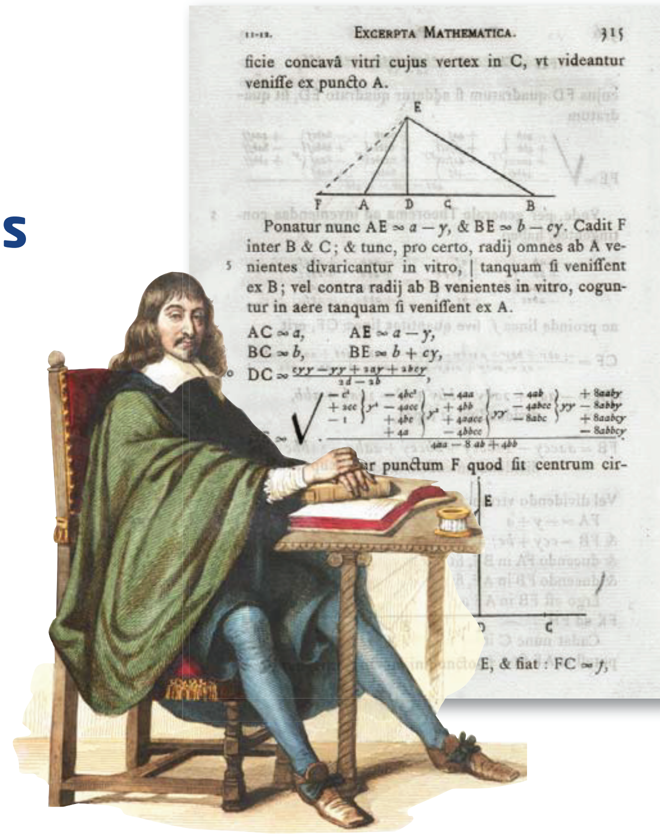
Which applications of nanotechnology have you used?

8.2

Solving Systems of Equations Algebraically

Focus on...

- modelling a situation using a system of linear-quadratic or quadratic-quadratic equations
- relating a system of linear-quadratic or quadratic-quadratic equations to a problem
- determining the solution of a system of linear-quadratic or quadratic-quadratic equations algebraically
- interpreting points of intersection of a system of linear-quadratic or quadratic-quadratic equations
- solving a problem that involves a system of linear-quadratic or quadratic-quadratic equations



René Descartes

Many ancient civilizations, such as Egyptian, Babylonian, Greek, Hindu, Chinese, Arabic, and European, helped develop the algebra we use today. Initially problems were stated and solved verbally or geometrically without the use of symbols. The French mathematician François Viète (1540–1603) popularized using algebraic symbols, but René Descartes' (1596–1650) thoroughly thought-out symbolism for algebra led directly to the notation we use today. Do you recognize the similarities and differences between his notation and ours?

Investigate Solving Systems of Equations Algebraically

1. Solve the following system of linear-quadratic equations graphically using graphing technology.
$y = x + 6$
$y = x^2$

2. a) How could you use the algebraic method of elimination or substitution to solve this system of equations?

b) What quadratic equation would you need to solve?

3. How are the roots of the quadratic equation in step 2b) related to the solution to the system of linear-quadratic equations?

4. Graph the related function for the quadratic equation from step 2b) in the same viewing window as the system of equations. Imagine a vertical line drawn through a solution to the system of equations in step 1. Where would this line intersect the equation from step 2b)? Explain this result.

5. What can you conclude about the relationship between the roots of the equation from step 2b) and the solution to the initial system of equations?

6. Consider the following system of quadratic-quadratic equations. Repeat steps 1 to 5 for this system.

$y = 2x^2 + 3x - 3$
$y = x^2 + x$

Reflect and Respond

7. Why are the x-coordinates in the solutions to the system of equations the same as the roots for the single equation you created using substitution or elimination? You may want to use sketches to help you explain.
8. Explain how you could solve a system of linear-quadratic or quadratic-quadratic equations without using any of the graphing steps in this investigation.

Link the Ideas

Recall from the previous section that systems of equations can have, depending on the type of system, 0, 1, 2, or infinite real solutions. You can apply the algebraic methods of substitution and elimination that you used to solve systems of linear equations to solve systems of linear-quadratic and quadratic-quadratic equations.

Why is it important to be able to solve systems algebraically as well as graphically?

Example 1

Solve a System of Linear-Quadratic Equations Algebraically

a) Solve the following system of equations.

$5x - y = 10$
$x^2 + x - 2y = 0$

b) Verify your solution.

Solution

a) Method 1: Use Substitution

Since the quadratic term is in the variable x, solve the linear equation for y.

Solve the linear equation for y.

$5x - y = 10$
$y = 5x - 10$

Why is it easier to solve the first equation for y?

Substitute $5x - 10$ for y in the quadratic equation and simplify.

$x^2 + x - 2y = 0$
$x^2 + x - 2(5x - 10) = 0$
$x^2 - 9x + 20 = 0$

Solve the quadratic equation by factoring.

$(x - 4)(x - 5) = 0$
$x = 4$ or $x = 5$

Substitute these values into the original linear equation to determine the corresponding values of y.

Why substitute into the linear equation rather than the quadratic?

When $x = 4$:	When $x = 5$:
$5x - y = 10$	$5x - y = 10$
$5(4) - y = 10$	$5(5) - y = 10$
$y = 10$	$y = 15$

The two solutions are (4, 10) and (5, 15).

Method 2: Use Elimination

Align the terms with the same degree.

Since the quadratic term is in the variable x, eliminate the y-term.

$$5x - y = 10 \quad ①$$
$$x^2 + x - 2y = 0 \quad ②$$

Multiply ① by -2 so that there is an opposite term to $-2y$ in ①.

$$-2(5x - y) = -2(10)$$
$$-10x + 2y = -20 \quad ③$$

Add ③ and ① to eliminate the y-terms.

$$\begin{array}{r} -10x + 2y = -20 \\ x^2 + x - 2y = 0 \\ \hline x^2 - 9x = -20 \end{array}$$

Then, solve the equation $x^2 - 9x + 20 = 0$ by factoring, as in the substitution method above, to obtain the two solutions (4, 10) and (5, 15).

What do the two solutions tell you about the appearance of the graphs of the two equations?

b) To verify the solutions, substitute each ordered pair into the original equations.

How could you verify the solutions using technology?

Verify the solution (4, 10):

Left Side	Right Side	Left Side	Right Side
$5x - y$	10	$x^2 + x - 2y$	0
$= 5(4) - 10$		$= 4^2 + 4 - 2(10)$	
$= 20 - 10$		$= 16 + 4 - 20$	
$= 10$		$= 0$	
Left Side = Right Side		Left Side = Right Side	

Verify the solution (5, 15):

Left Side	Right Side	Left Side	Right Side
$5x - y$	10	$x^2 + x - 2y$	0
$= 5(5) - 15$		$= 5^2 + 5 - 2(15)$	
$= 25 - 15$		$= 25 + 5 - 30$	
$= 10$		$= 0$	
Left Side = Right Side		Left Side = Right Side	

Both solutions are correct.

The two solutions are (4, 10) and (5, 15).

Your Turn

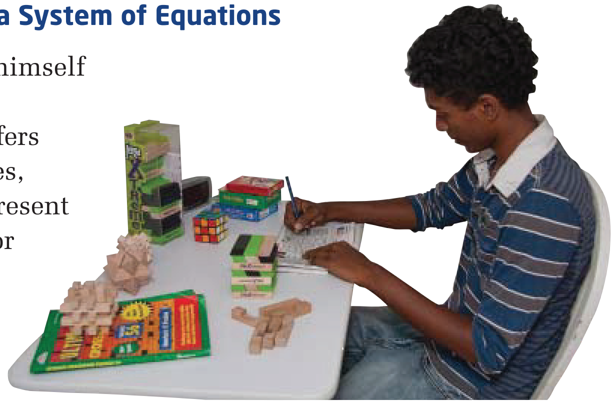
Solve the following system of equations algebraically.

$$3x + y = -9$$
$$4x^2 - x + y = -9$$

Example 2

Model a Situation With a System of Equations

Glen loves to challenge himself with puzzles. He comes across a Web site that offers online interactive puzzles, but the puzzle-makers present the following problem for entry to their site.



So you like puzzles? Well, prove your worthiness by solving this conundrum.

Determine two integers such that the sum of the smaller number and twice the larger number is 46. Also, when the square of the smaller number is decreased by three times the larger, the result is 93. In the box below, enter the smaller number followed by the larger number and you will gain access to our site.

a) Write a system of equations that relates to the problem.

b) Solve the system algebraically. What is the code that gives access to the site?

Solution

a) Let S represent the smaller number.
Let L represent the larger number.
Use these variables to write an equation to represent the first statement: "the sum of the smaller number and twice the larger number is 46."
$S + 2L = 46$

Next, write an equation to represent the second statement: "when the square of the smaller number is decreased by three times the larger, the result is 93."
$S^2 - 3L = 93$

Solving the system of equations gives the numbers that meet both sets of conditions.

b) Use the elimination method.

Why was the elimination method chosen? Could you use the substitution method instead?

$S + 2L = 46$ ①

$S^2 - 3L = 93$ ②

Multiply ① by 3 and ② by 2.

$3(S + 2L) = 3(46)$

$3S + 6L = 138$ ③

$2(S^2 - 3L) = 2(93)$

$2S^2 - 6L = 186$ ④

Add ③ and ④ to eliminate L.

$$\begin{array}{r} 3S + 6L = 138 \\ 2S^2 - 6L = 186 \\ \hline 2S^2 + 3S = 324 \end{array}$$

Why can you not eliminate the variable S?

Solve $2S^2 + 3S - 324 = 0$.

Factor.

$(2S + 27)(S - 12) = 0$

$S = -13.5$ or $S = 12$

Since the numbers are supposed to be integers, $S = 12$.

Substitute $S = 12$ into the linear equation to determine the value of L.

Why was ① chosen to substitute into?

$S + 2L = 46$

$12 + 2L = 46$

$2L = 34$

$L = 17$

The solution is (12, 17).

Verify the solution by substituting (12, 17) into the original equations:

Left Side	Right Side	Left Side	Right Side
$S + 2L$	46	$S^2 - 3L$	93
$= 12 + 2(17)$		$= 12^2 - 3(17)$	
$= 12 + 34$		$= 144 - 51$	
$= 46$		$= 93$	
Left Side = Right Side		Left Side = Right Side	

The solution is correct.

The two numbers for the code are 12 and 17. The access key is 1217.

Your Turn

Determine two integers that have the following relationships: Fourteen more than twice the first integer gives the second integer. The second integer increased by one is the square of the first integer.

a) Write a system of equations that relates to the problem.

b) Solve the system algebraically.

Example 3

Solve a Problem Involving a Linear-Quadratic System

A Canadian cargo plane drops a crate of emergency supplies to aid-workers on the ground. The crate drops freely at first before a parachute opens to bring the crate gently to the ground. The crate's height, h, in metres, above the ground t seconds after leaving the aircraft is given by the following two equations.

$h = -4.9t^2 + 700$ represents the height of the crate during free fall.

$h = -5t + 650$ represents the height of the crate with the parachute open.

a) How long after the crate leaves the aircraft does the parachute open? Express your answer to the nearest hundredth of a second.
b) What height above the ground is the crate when the parachute opens? Express your answer to the nearest metre.
c) Verify your solution.

Solution

a) The moment when the parachute opens corresponds to the point of intersection of the two heights. The coordinates can be determined by solving a system of equations.

The linear equation is written in terms of the variable h, so use the method of substitution.

Substitute $-5t + 650$ for h in the quadratic equation.

$$h = -4.9t^2 + 700$$
$$-5t + 650 = -4.9t^2 + 700$$
$$4.9t^2 - 5t - 50 = 0$$

Solve using the quadratic formula.

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$t = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(4.9)(-50)}}{2(4.9)}$$
$$t = \frac{5 \pm \sqrt{1005}}{9.8}$$
$$t = \frac{5 + \sqrt{1005}}{9.8} \quad \text{or} \quad t = \frac{5 - \sqrt{1005}}{9.8}$$
$$t = 3.745\ldots \quad \text{or} \quad t = -2.724\ldots$$

Why is $t = -2.724...$ rejected as a solution to this problem?

The parachute opens about 3.75 s after the crate leaves the plane.

b) To find the crate's height above the ground, substitute the value $t = 3.745...$ into the linear equation.

$h = -5t + 650$
$h = -5(3.745...) + 650$
$h = 631.274...$

The crate is about 631 m above the ground when the parachute opens.

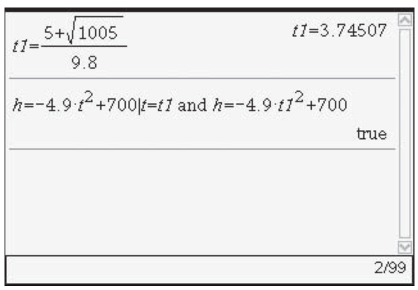
c) **Method 1: Use Paper and Pencil**

To verify the solution, substitute the answer for t into the first equation of the system.

$h = -4.9t^2 + 700$
$h = -4.9(3.745...)^2 + 700$
$h = 631.274...$

The solution is correct.

Method 2: Use Technology



The solution is correct.

The crate is about 631 m above the ground when the parachute opens.

Your Turn

Suppose the crate's height above the ground is given by the following two equations.

$h = -4.9t^2 + 900$
$h = -4t + 500$

a) How long after the crate leaves the aircraft does the parachute open? Express your answer to the nearest hundredth of a second.

b) What height above the ground is the crate when the parachute opens? Express your answer to the nearest metre.

c) Verify your solution.

Example 4

Solve a System of Quadratic-Quadratic Equations Algebraically

a) Solve the following system of equations.

$3x^2 - x - y - 2 = 0$

$6x^2 + 4x - y = 4$

b) Verify your solution.

Solution

a) Both equations contain a single y-term, so use elimination.

$3x^2 - x - y = 2$ ①

$6x^2 + 4x - y = 4$ ②

Why can x not be eliminated?

Could this system be solved by substitution? Explain.

Subtract ① from ② to eliminate y.

$$\begin{array}{r} 6x^2 + 4x - y = 4 \\ 3x^2 - x - y = 2 \\ \hline 3x^2 + 5x = 2 \end{array}$$

Solve the quadratic equation.

$$\begin{aligned} 3x^2 + 5x &= 2 \\ 3x^2 + 5x - 2 &= 0 \\ 3x^2 + 6x - x - 2 &= 0 \\ 3x(x + 2) - 1(x + 2) &= 0 \\ (x + 2)(3x - 1) &= 0 \end{aligned}$$

Factor by grouping.

$x = -2$ or $x = \frac{1}{3}$

Substitute these values into the equation $3x^2 - x - y = 2$ to determine the corresponding values of y.

When $x = -2$:

$$\begin{aligned} 3x^2 - x - y &= 2 \\ 3(-2)^2 - (-2) - y &= 2 \\ 12 + 2 - y &= 2 \\ y &= 12 \end{aligned}$$

When $x = \frac{1}{3}$:

$$\begin{aligned} 3x^2 - x - y &= 2 \\ 3\left(\frac{1}{3}\right)^2 - \frac{1}{3} - y &= 2 \\ \frac{1}{3} - \frac{1}{3} - y &= 2 \\ y &= -2 \end{aligned}$$

The system has two solutions: $(-2, 12)$ and $\left(\frac{1}{3}, -2\right)$.

What do the two solutions tell you about the appearance of the graphs of the two equations?

b) To verify the solutions, substitute each ordered pair into the original equations.

Verify the solution (−2, 12):

Left Side	Right Side
$3x^2 - x - y - 2$	0
$= 3(-2)^2 - (-2) - 12 - 2$	
$= 12 + 2 - 12 - 2$	
$= 0$	

Left Side = Right Side

Left Side	Right Side
$6x^2 + 4x - y$	4
$= 6(-2)^2 + 4(-2) - 12$	
$= 24 - 8 - 12$	
$= 4$	

Left Side = Right Side

Verify the solution $\left(\frac{1}{3}, -2\right)$:

Left Side	Right Side
$3x^2 - x - y - 2$	0
$= 3\left(\frac{1}{3}\right)^2 - \left(\frac{1}{3}\right) - (-2) - 2$	
$= \frac{1}{3} - \frac{1}{3} + 2 - 2$	
$= 0$	

Left Side = Right Side

Left Side	Right Side
$6x^2 + 4x - y$	4
$= 6\left(\frac{1}{3}\right)^2 + 4\left(\frac{1}{3}\right) - (-2)$	
$= \frac{2}{3} + \frac{4}{3} + 2$	
$= 4$	

Left Side = Right Side

Both solutions are correct.

The system has two solutions: (−2, 12) and $\left(\frac{1}{3}, -2\right)$.

Your Turn

a) Solve the system algebraically. Explain why you chose the method that you did.

$6x^2 - x - y = -1$

$4x^2 - 4x - y = -6$

b) Verify your solution.

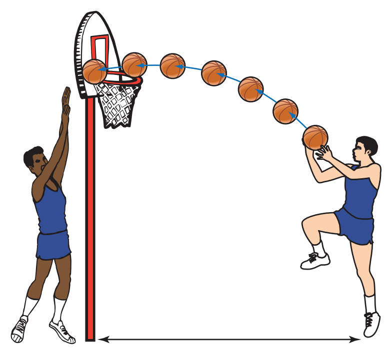
Example 5

Solve a Problem Involving a Quadratic-Quadratic System

During a basketball game, Ben completes an impressive "alley-oop." From one side of the hoop, his teammate Luke lobs a perfect pass toward the basket. Directly across from Luke, Ben jumps up, catches the ball and tips it into the basket. The path of the ball thrown by Luke can be modelled by the equation $d^2 - 2d + 3h = 9$, where d is the horizontal distance of the ball from the centre of the hoop, in metres, and h is the height of the ball above the floor, in metres. The path of Ben's jump can be modelled by the equation $5d^2 - 10d + h = 0$, where d is his horizontal distance from the centre of the hoop, in metres, and h is the height of his hands above the floor, in metres.

a) Solve the system of equations algebraically. Give your solution to the nearest hundredth.

b) Interpret your result. What assumptions are you making?



Did You Know?

In 1891 at a small college in Springfield Massachusetts, a Canadian physical education instructor named James Naismith, invented the game of basketball as a way to keep his students active during winter months. The first game was played with a soccer ball and two peach baskets, with numerous stoppages in play to manually retrieve the ball from the basket.

Solution

a) The system to solve is

$d^2 - 2d + 3h = 9$

$5d^2 - 10d + h = 0$

Solve the second equation for h since the leading coefficient of this term is 1.

$h = -5d^2 + 10d$

Substitute $-5d^2 + 10d$ for h in the first equation.

$$\begin{aligned} d^2 - 2d + 3h &= 9 \\ d^2 - 2d + 3(-5d^2 + 10d) &= 9 \\ d^2 - 2d - 15d^2 + 30d &= 9 \\ 14d^2 - 28d + 9 &= 0 \end{aligned}$$

Solve using the quadratic formula.

$$d = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$d = \frac{-(-28) \pm \sqrt{(-28)^2 - 4(14)(9)}}{2(14)}$$
$$d = \frac{28 \pm \sqrt{280}}{28}$$
$$d = \frac{14 + \sqrt{70}}{14} \quad \text{or} \quad d = \frac{14 - \sqrt{70}}{14}$$
$$d = 1.597\ldots \qquad d = 0.402\ldots$$

Substitute these values of d into the equation $h = -5d^2 + 10d$ to find the corresponding values of h.

For $d = \frac{14 + \sqrt{70}}{14}$:
$$h = -5d^2 + 10d$$
$$h = -5\left(\frac{14 + \sqrt{70}}{14}\right)^2 + 10\left(\frac{14 + \sqrt{70}}{14}\right)$$
$$h = 3.214\ldots$$

For $d = \frac{14 - \sqrt{70}}{14}$:
$$h = -5d^2 + 10d$$
$$h = -5\left(\frac{14 - \sqrt{70}}{14}\right)^2 + 10\left(\frac{14 - \sqrt{70}}{14}\right)$$
$$h = 3.214\ldots$$

To the nearest hundredth, the solutions to the system are (0.40, 3.21) and (1.60, 3.21).

How can you verify the solutions?

b) The parabolic path of the ball and Ben's parabolic path will intersect at two locations: at a distance of 0.40 m from the basket and at a distance of 1.60 m from the basket, in both cases at a height of 3.21 m. Ben will complete the alley-oop if he catches the ball at the distance of 0.40 m from the hoop. The ball is at the same height, 3.21 m, on its upward path toward the net but it is still 1.60 m away.

Why is the solution of 1.60 m not appropriate in this context?

This will happen if you assume Ben times his jump appropriately, is physically able to make the shot, and the shot is not blocked by another player.

Your Turn

Terri makes a good hit and the baseball travels on a path modelled by $h = -0.1x^2 + 2x$. Ruth is in the outfield directly in line with the path of the ball. She runs toward the ball and jumps to try to catch it. Her jump is modelled by the equation $h = -x^2 + 39x - 378$. In both equations, x is the horizontal distance in metres from home plate and h is the height of the ball above the ground in metres.

a) Solve the system algebraically. Round your answer to the nearest hundredth.

b) Explain the meaning of the point of intersection. What assumptions are you making?

Key Ideas

- Solve systems of linear-quadratic or quadratic-quadratic equations algebraically by using either a substitution method or an elimination method.
- To solve a system of equations in two variables using substitution,
  - isolate one variable in one equation
  - substitute the expression into the other equation and solve for the remaining variable
  - substitute the value(s) into one of the original equations to determine the corresponding value(s) of the other variable
  - verify your answer by substituting into both original equations
- To solve a system of equations in two variables using elimination,
  - if necessary, rearrange the equations so that the like terms align
  - if necessary, multiply one or both equations by a constant to create equivalent equations with a pair of variable terms with opposite coefficients
  - add or subtract to eliminate one variable and solve for the remaining variable
  - substitute the value(s) into one of the original equations to determine the corresponding value(s) of the other variable
  - verify your answer(s) by substituting into both original equations

Check Your Understanding

Practise

Where necessary, round your answers to the nearest hundredth.

1. Verify that (5, 7) is a solution to the following system of equations.

$k + p = 12$
$4k^2 - 2p = 86$

2. Verify that $\left(\frac{1}{3}, \frac{3}{4}\right)$ is a solution to the following system of equations.

$18w^2 - 16z^2 = -7$
$144w^2 + 48z^2 = 43$

3. Solve each system of equations by substitution, and verify your solution(s).

a) $x^2 - y + 2 = 0$
$4x = 14 - y$

b) $2x^2 - 4x + y = 3$
$4x - 2y = -7$

c) $7d^2 + 5d - t - 8 = 0$
$10d - 2t = -40$

d) $3x^2 + 4x - y - 8 = 0$
$y + 3 = 2x^2 + 4x$

e) $y + 2x = x^2 - 6$
$x + y - 3 = 2x^2$

4. Solve each system of equations by elimination, and verify your solution(s).

a) $6x^2 - 3x = 2y - 5$
$2x^2 + x = y - 4$

b) $x^2 + y = 8x + 19$
$x^2 - y = 7x - 11$

c) $2p^2 = 4p - 2m + 6$
$5m + 8 = 10p + 5p^2$

d) $9w^2 + 8k = -14$
$w^2 + k = -2$

e) $4h^2 - 8t = 6$
$6h^2 - 9 = 12t$

5. Solve each system algebraically. Explain why you chose the method you used.

a) $y - 1 = -\frac{7}{8}x$
$3x^2 + y = 8x - 1$

b) $8x^2 + 5y = 100$
$6x^2 - x - 3y = 5$

c) $x^2 - \frac{48}{9}x + \frac{1}{3}y + \frac{1}{3} = 0$
$-\frac{5}{4}x^2 - \frac{3}{2}x + \frac{1}{4}y - \frac{1}{2} = 0$

Apply

6. Alex and Kaela are considering the two equations $n - m^2 = 7$ and $2m^2 - 2n = -1$. Without making any calculations, they both claim that the system involving these two equations has no solution.

Alex's reasoning:
If I double every term in the first equation and then use the elimination method, both of the variables will disappear, so the system does not have a solution.

Kaela's reasoning:
If I solve the first equation for n and substitute into the second equation, I will end up with an equation without any variables, so the system does not have a solution.

a) Is each person's reasoning correct?

b) Verify the conclusion graphically.

7. Marie-Soleil solved two systems of equations using elimination. Instead of creating opposite terms and adding, she used a subtraction method. Her work for the elimination step in two different systems of equations is shown below.

First System

$$\begin{array}{rcl} 5x + 2y &=& 12 \\ x^2 - 2x + 2y &=& 7 \\ \hline -x^2 + 7x &=& 5 \end{array}$$

Second System

$$\begin{array}{rcl} 12m^2 - 4m - 8n &=& -3 \\ 9m^2 - m - 8n &=& 2 \\ \hline 3m^2 - 3m &=& -5 \end{array}$$

a) Study Marie-Soleil's method. Do you think this method works? Explain.

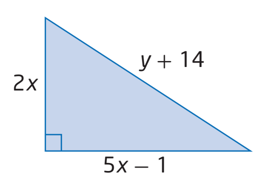
b) Redo the first step in each system by multiplying one of the equations by -1 and adding. Did you get the same results as Marie-Soleil?

c) Do you prefer to add or subtract to eliminate a variable? Explain why.

8. Determine the values of m and n if (2, 8) is a solution to the following system of equations.

$mx^2 - y = 16$
$mx^2 + 2y = n$

9. The perimeter of the right triangle is 60 m. The area of the triangle is $10y$ square metres.



a) Write a simplified expression for the triangle's perimeter in terms of x and y.

b) Write a simplified expression for the triangle's area in terms of x and y.

c) Write a system of equations and explain how it relates to this problem.

d) Solve the system for x and y. What are the dimensions of the triangle?

e) Verify your solution.

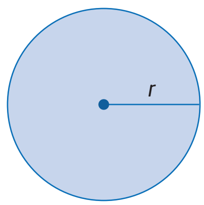
10. Two integers have a difference of -30. When the larger integer is increased by 3 and added to the square of the smaller integer, the result is 189.

a) Model the given information with a system of equations.

b) Determine the value of the integers by solving the system.

c) Verify your solution.

11. The number of centimetres in the circumference of a circle is three times the number of square centimetres in the area of the circle.



a) Write the system of linear-quadratic equations, in two variables, that models the circle with the given property.

b) What are the radius, circumference, and area of the circle with this property?

12. A 250-g ball is thrown into the air with an initial velocity of 22.36 m/s. The kinetic energy, E_k, of the ball is given by the equation $E_k = \frac{5}{32}(d - 20)^2$ and its potential energy, E_p, is given by the equation $E_p = -\frac{5}{32}(d - 20)^2 + 62.5$, where energy is measured in joules (J) and d is the horizontal distance travelled by the ball, in metres.

a) At what distances does the ball have the same amount of kinetic energy as potential energy?

b) How many joules of each type of energy does the ball have at these distances?

c) Verify your solution by graphing.

d) When an object is thrown into the air, the total mechanical energy of the object is the sum of its kinetic energy and its potential energy. On Earth, one of the properties of an object in motion is that the total mechanical energy is a constant. Does the graph of this system show this property? Explain how you could confirm this observation.

13. The 2015-m-tall Mount Asgard in Auyuittuq (ow you eet took) National Park, Baffin Island, Nunavut, was used in the opening scene for the James Bond movie *The Spy Who Loved Me*. A stuntman skis off the edge of the mountain, free-falls for several seconds, and then opens a parachute. The height, h, in metres, of the stuntman above the ground t seconds after leaving the edge of the mountain is given by two functions.

$h(t) = -4.9t^2 + 2015$ represents the height of the stuntman before he opens the parachute.

$h(t) = -10.5t + 980$ represents the height of the stuntman after he opens the parachute.

a) For how many seconds does the stuntman free-fall before he opens his parachute?

b) What height above the ground was the stuntman when he opened the parachute?

c) Verify your solutions.

Did You Know?

Mount Asgard, named after the kingdom of the gods in Norse mythology, is known as Sivanitirutinguak (see va kneek tea goo ting goo ak) to Inuit. This name, in Inuktitut, means "shape of a bell."

14. A table of values is shown for two different quadratic functions.

First Quadratic

x	y
−1	2
0	0
1	2
2	8

Second Quadratic

x	y
−5	4
−4	1
−3	0
−2	1

a) Use paper and pencil to plot each set of ordered pairs on the same grid. Sketch the quadratic functions.

b) Estimate the solution to the system involving the two quadratic functions.

c) Determine a quadratic equation for each function and model a quadratic-quadratic system with these equations.

d) Solve the system of equations algebraically. How does your solution compare to your estimate in part b)?

15. When a volcano erupts, it sends lava fragments flying through the air. From the point where a fragment is blasted into the air, its height, h, in metres, relative to the horizontal distance travelled, x, in metres, can be approximated using the function $h(x) = -\frac{4.9}{(v_0 \cos \theta)^2}x^2 + (\tan \theta)x + h_0$, where v_0 is the initial velocity of the fragment, in metres per second; θ is the angle, in degrees, relative to the horizontal at which the fragment starts its path; and h_0 is the initial height, in metres, of the fragment.

a) The height of the summit of a volcano is 2500 m. If a lava fragment blasts out of the middle of the summit at an angle of 45° travelling at 60 m/s, confirm that the function $h(x) = -0.003x^2 + x + 2500$ approximately models the fragment's height relative to the horizontal distance travelled. Confirm that a fragment blasted out at an angle of 60° travelling at 60 m/s can be approximately modelled by the function $h(x) = -0.005x^2 + 1.732x + 2500$.

b) Solve the system

$h(x) = -0.003x^2 + x + 2500$
$h(x) = -0.005x^2 + 1.732x + 2500$

c) Interpret your solution and the conditions required for it to be true.

Did You Know?

Iceland is one of the most active volcanic areas in the world. On April 14, 2010, when Iceland's Eyjafjallajokull (ay yah fyah lah yoh kuul) volcano had a major eruption, ash was sent high into Earth's atmosphere and drifted south and east toward Europe. This large ash cloud wreaked havoc on air traffic. In fear that the airplanes' engines would be clogged by the ash, thousands of flights were cancelled for many days.

16. In western Canada, helicopter "bombing" is used for avalanche control. In high-risk areas, explosives are dropped onto the mountainside to safely start an avalanche. The function $h(x) = -\frac{5}{1600}x^2 + 200$ represents the height, h, in metres, of the explosive once it has been thrown from the helicopter, where x is the horizontal distance, in metres, from the base of the mountain. The mountainside is modelled by the function $h(x) = 1.19x$.

a) How can the following system of equations be used for this scenario?

$h = -\frac{5}{1600}x^2 + 200$
$h = 1.19x$

b) At what height up the mountain does the explosive charge land?

17. The monthly economic situation of a manufacturing firm is given by the following equations.

$$R = 5000x - 10x^2$$
$$R_M = 5000 - 20x$$
$$C = 300x + \frac{1}{12}x^2$$
$$C_M = 300 + \frac{1}{4}x^2$$

where x represents the quantity sold, R represents the firm's total revenue, R_M represents marginal revenue, C represents total cost, and C_M represents the marginal cost. All costs are in dollars.

a) Maximum profit occurs when marginal revenue is equal to marginal cost. How many items should be sold to maximize profit?

b) Profit is total revenue minus total cost. What is the firm's maximum monthly profit?

Extend

18. Kate is an industrial design engineer. She is creating the program for cutting fabric for a shade sail. The shape of a shade sail is defined by three intersecting parabolas. The equations of the parabolas are

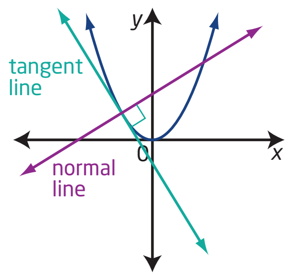
$$y = x^2 + 8x + 16$$
$$y = x^2 - 8x + 16$$
$$y = -\frac{x^2}{8} + 2$$

where x and y are measurements in metres.

a) Use an algebraic method to determine the coordinates of the three vertices of the sail.

b) Estimate the area of material required to make the sail.

19. A normal line is a line that is perpendicular to a tangent line of a curve at the point of tangency.



The line $y = 4x - 2$ is tangent to the curve $y = 2x^2 - 4x + 6$ at a point A.

a) What are the coordinates of point A?

b) What is the equation of the normal line to the curve $y = 2x^2 - 4x + 6$ at the point A?

c) The normal line intersects the curve again at point B, creating chord AB. Determine the length of this chord.

20. Solve the following system of equations using an algebraic method.

$$y = \frac{2x - 1}{x}$$
$$\frac{x}{x + 2} + y - 2 = 0$$

21. Determine the equations for the linear-quadratic system with the following properties. The vertex of the parabola is at (−1, −4.5). The line intersects the parabola at two points. One point of intersection is on the y-axis at (0, −4) and the other point of intersection is on the x-axis.

Create Connections

22. Consider graphing methods and algebraic methods for solving a system of equations. What are the advantages and the disadvantages of each method? Create your own examples to model your answer.

23. A parabola has its vertex at (−3, −1) and passes through the point (−2, 1). A second parabola has its vertex at (−1, 5) and has y-intercept 4. What are the approximate coordinates of the point(s) at which these parabolas intersect?

24. Use algebraic reasoning to show that the graphs of $y = -\frac{1}{2}x - 2$ and $y = x^2 - 4x + 2$ do not intersect.

25. **MINI LAB** In this activity, you will explore the effects that varying the parameters b and m in a linear equation have on a system of linear-quadratic equations.

Step 1 Consider the system of linear-quadratic equations $y = x^2$ and $y = x + b$, where $b \in R$. Graph the system of equations for different values of b. Experiment with changing the value of b so that for some of your values of b the parabola and the line intersect in two points, and not for others. For what value of b do the parabola and the line intersect in exactly one point? Based on your results, predict the values of b for which the system has two real solutions, one real solution, and no real solution.

Step 2 Algebraically determine the values of b for which the system has two real solutions, one real solution, and no real solution.

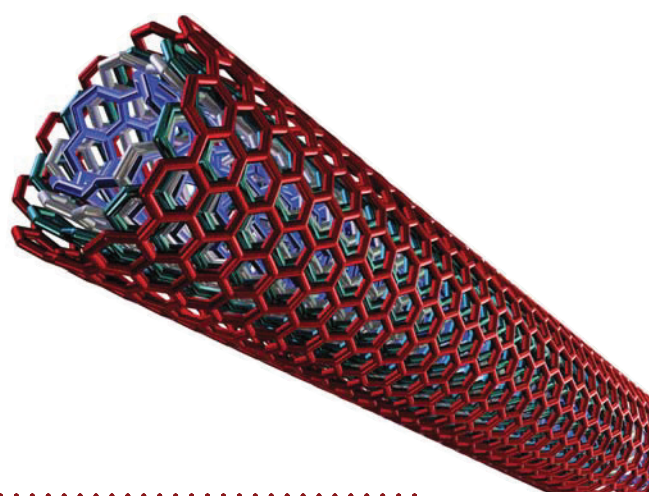
Step 3 Consider the system of linear-quadratic equations $y = x^2$ and $y = mx - 1$, where $m \in R$. Graph the system of equations for different values of m. Experiment with changing the value of m. For what value of m do the parabola and the line intersect in exactly one point? Based on your results, predict the values of m for which the system has two real solutions, one real solution, and no real solution.

Step 4 Algebraically determine the values of m for which the system has two real solutions, one real solution and no real solution.

Step 5 Consider the system of linear-quadratic equations $y = x^2$ and $y = mx + b$, where $m, b \in R$. Determine the conditions on m and b for which the system has two real solutions, one real solution, and no real solution.

Project Corner — Carbon Nanotubes and Engineering

- Carbon nanotubes are cylindrical molecules made of carbon. They have many amazing properties and, as a result, can be used in a number of different applications.
- Carbon nanotubes are up to 100 times stronger than steel and only $\frac{1}{16}$ its mass.
- Researchers mix nanotubes with plastics as reinforcers.



Nanotechnology is already being applied to sports equipment such as bicycles, golf clubs, and tennis rackets. Future uses will include things like aircraft, bridges, and cars. What are some other things that could be enhanced by this stronger and lighter product?

Chapter 8 Review

8.1 Solving Systems of Equations Graphically, pages 424–439

Where necessary, round your answers to the nearest hundredth.

1. Consider the tables of values for $y = -1.5x - 2$ and $y = -2(x - 4)^2 + 3$.

x	y
0.5	−2.75
1	−3.5
1.5	−4.25
2	−5
2.5	−5.75
3	−6.5

x	y
0.5	−21.5
1	−15
1.5	−9.5
2	−5
2.5	−1.5
3	1

a) Use the tables to determine a solution to the system of equations

$y = -1.5x - 2$
$y = -2(x - 4)^2 + 3$

b) Verify this solution by graphing.

c) What is the other solution to the system?

2. State the number of possible solutions to each system. Include sketches to support your answers.

a) a system involving a parabola and a horizontal line

b) a system involving two parabolas that both open upward

c) a system involving a parabola and a line with a positive slope

3. Solve each system of equations by graphing.

a) $y = \frac{2}{3}x + 4$
$y = -3(x + 6)^2$

b) $y = x^2 - 4x + 1$
$y = -\frac{1}{2}(x - 2)^2 + 3$

4. Adam graphed the system of quadratic equations $y = x^2 + 1$ and $y = x^2 + 3$ on a graphing calculator. He speculates that the two graphs will intersect at some large value of y. Is Adam correct? Explain.

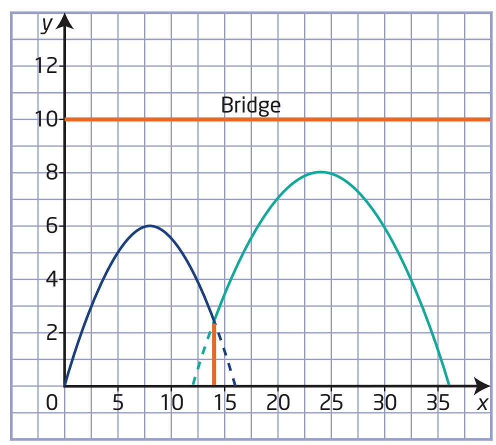
5. Solve each system of equations by graphing.

a) $p = \frac{1}{3}(x + 2)^2 + 2$
$p = \frac{1}{3}(x - 1)^2 + 3$

b) $y = -6x^2 - 4x + 7$
$y = x^2 + 2x - 6$

c) $t = -3d^2 - 2d + 3.25$
$t = \frac{1}{8}d - 5$

6. An engineer constructs side-by-side parabolic arches to support a bridge over a road and a river. The arch over the road has a maximum height of 6 m and a width of 16 m. The river arch has a maximum height of 8 m, but its width is reduced by 4 m because it intersects the arch over the road. Without this intersection, the river arch would have a width of 24 m. A support footing is used at the intersection point of the arches. The engineer sketched the arches on a coordinate system. She placed the origin at the left most point of the road.



a) Determine the equation that models each arch.

b) Solve the system of equations.

c) What information does the solution to the system give the engineer?

7. Caitlin is at the base of a hill with a constant slope. She kicks a ball as hard as she can up the hill.

   a) Explain how the following system models this situation.

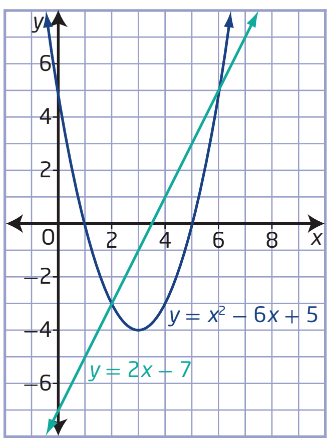
   $h = -0.09d^2 + 1.8d$

   $h = \frac{1}{2}d$

   b) Solve the system.

   c) Interpret the point(s) of intersection in the context.

8.2 Solving Systems of Equations Algebraically, pages 440–456

8. 

   a) Estimate the solutions to the system of equations shown in the graph.

   b) Solve the system algebraically.

9. Without solving the system $4m^2 - 3n = -2$ and $m^2 + \frac{7}{2}m + 5n = 7$, determine which solution is correct: $\left(\frac{1}{2}, 1\right)$ or $\left(\frac{1}{2}, -1\right)$.

10. Solve each system algebraically, giving exact answers. Explain why you chose the method you used.

   a) $p = 3k + 1$
   $p = 6k^2 + 10k - 4$

   b) $4x^2 + 3y = 1$
   $3x^2 + 2y = 4$

   c) $\frac{w^2}{2} + \frac{w}{4} - \frac{z}{2} = 3$
   $\frac{w^2}{3} - \frac{3w}{4} + \frac{z}{6} + \frac{1}{3} = 0$

   d) $2y - 1 = x^2 - x$
   $x^2 + 2x + y - 3 = 0$

11. The approximate height, h, in metres, travelled by golf balls hit with two different clubs over a horizontal distance of d metres is given by the following functions:

   seven-iron: $h(d) = -0.002d^2 + 0.3d$
   nine-iron: $h(d) = -0.004d^2 + 0.5d$

   a) At what distances is the ball at the same height when either of the clubs is used?

   b) What is this height?

12. Manitoba has many biopharmaceutical companies. Suppose scientists at one of these companies grow two different cell cultures in an identical nutrient-rich medium. The rate of increase, S, in square millimetres per hour, of the surface area of each culture after t hours is modelled by the following quadratic functions:

   First culture: $S(t) = -0.007t^2 + 0.05t$
   Second culture: $S(t) = -0.0085t^2 + 0.06t$

   a) What information would the scientists gain by solving the system of related equations?

   b) Solve the system algebraically.

   c) Interpret your solution.

Chapter 8 Practice Test

Multiple Choice

For #1 to #5, choose the best answer.

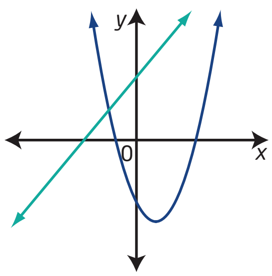
1. The graph for a system of equations is shown. In which quadrant(s) is there a solution to the system?

A I only

B II only

C I and II only

D II and III only



2. The system $y = \frac{1}{2}(x - 6)^2 + 2$ and $y = 2x + k$ has no solution. How many solutions does the system $y = -\frac{1}{2}(x - 6)^2 + 2$ and $y = 2x + k$ have?

A none

B one

C two

D infinitely many

3. Tables of values are shown for two different quadratic functions. What conclusion can you make about the related system of equations?

x	y
1	6
2	−3
3	−6
4	−3
5	6

x	y
1	−6
2	−3
3	−2
4	−3
5	−6

A It does not have a solution.

B It has at least two real solutions.

C It has an infinite number of solutions.

D It is quadratic-quadratic with a common vertex.

4. What is the solution to the following system of equations?

$y = (x + 2)^2 - 2$
$y = \frac{1}{2}(x + 2)^2$

A no solution

B $x = 2$

C $x = -4$ and $x = 2$

D $x = -4$ and $x = 0$

5. Connor used the substitution method to solve the system

$5m - 2n = 25$
$3m^2 - m + n = 10$

Below is Connor's solution for m. In which line did he make an error?

Connor's solution:
Solve the second equation for n:
$n = 10 - 3m^2 + m$ line 1
Substitute into the first equation:
$5m - 2(10 - 3m^2 + m) = 25$ line 2
$5m - 20 + 6m^2 - 2m = 25$
$6m^2 + 3m - 45 = 0$ line 3
$2m^2 + m - 15 = 0$
$(2m + 5)(m - 3) = 0$ line 4
$m = 2.5$ or $m = -3$

A line 1 **B** line 2

C line 3 **D** line 4

Short Answer

Where necessary, round your answers to the nearest hundredth.

6. A student determines that one solution to a system of quadratic-quadratic equations is (2, 1). What is the value of n if the equations are

$4x^2 - my = 10$
$mx^2 + ny = 20$

7. Solve algebraically.

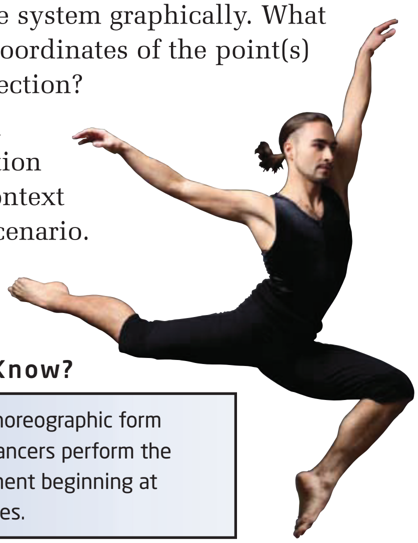
a) $5x^2 + 3y = -3 - x$
$2x^2 - x = -4 - 2y$

b) $y = 7x - 11$
$5x^2 - 3x - y = 6$

8. For a dance routine, the choreographer has arranged for two dancers to perform jeté jumps in canon. Sophie leaps first, and one count later Noah starts his jump. Sophie's jump can by modelled by the equation $h = -4.9t^2 + 5.1t$ and Noah's by the equation $h = -4.9(t - 0.5)^2 + 5.3(t - 0.5)$. In both equations, t is the time in seconds and h is the height in metres.

   a) Solve the system graphically. What are the coordinates of the point(s) of intersection?

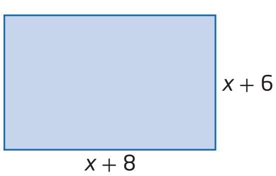
   b) Interpret the solution in the context of this scenario.



Did You Know?

Canon is a choreographic form where the dancers perform the same movement beginning at different times.

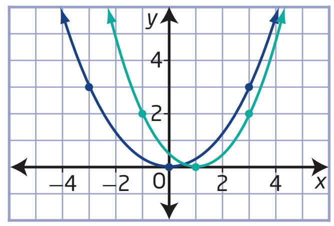
9. The perimeter of the rectangle is represented by $8y$ metres and the area is represented by $(6y + 3)$ square metres.



   a) Write two equations in terms of x and y: one for the perimeter and one for the area of the rectangle.

   b) Determine the perimeter and the area.

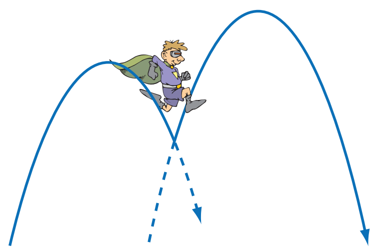
10. **a)** Determine a system of quadratic equations for the functions shown.



   b) Solve the system algebraically.

Extended Response

11. Computer animators design game characters to have many different abilities. The double-jump mechanic allows the character to do a second jump while in mid-air and change its first trajectory to a new one.

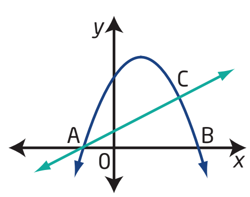


During a double jump, the first part of the jump is modelled by the equation $h = -12.8d^2 + 6.4d$, and the second part is modelled by the equation $h = -\frac{248}{15}(d - 0.7)^2 + 2$. In both equations, d is the horizontal distance and h is the height, in centimetres.

   a) Solve the system of quadratic-quadratic equations by graphing.

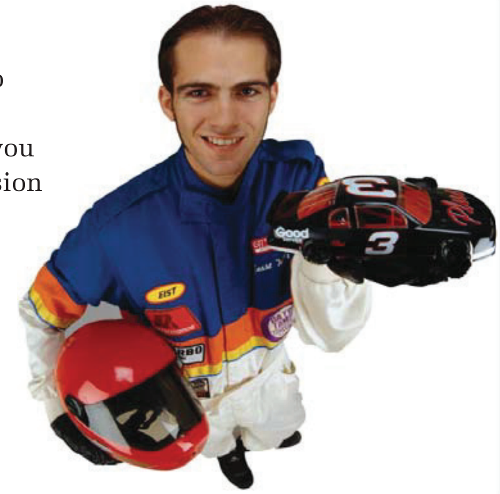
   b) Interpret your solution.

12. The parabola $y = -x^2 + 4x + 26.5$ intersects the x-axis at points A and B. The line $y = 1.5x + 5.25$ intersects the parabola at points A and C. Determine the approximate area of $\triangle$ABC.



Unit 4 Project

Nanotechnology



This part of your project will require you to be creative and to use your math skills. Combining your knowledge of parabolas and quadratic systems with the nanotechnology information you have gathered in this chapter, you will design a futuristic version of a every-day object. The object should have some linear and parabolic design lines.

Chapter 8 Task

Choose an object that you feel could be improved using nanotechnology. Look at the information presented in this chapter's Project Corners to give you ideas.

- Explain how the object you have chosen will be enhanced by using nanotechnology.
- Create a new design for your chosen object. Your design must include intersections of parabolic and linear design curves.
- Your design will inevitably go through a few changes as you develop it. Keep a well-documented record of the evolution of your design.
- Select a part of your design that involves an intersection of parabolas or an intersection of parabolas and lines. Determine model equations for each function involved in this part of your design.
- Using these equations, determine any points of intersection.
- What is the relevance of the points of intersection to the design of the object? How is it helpful to have model equations and to know the coordinates of the points of intersection?