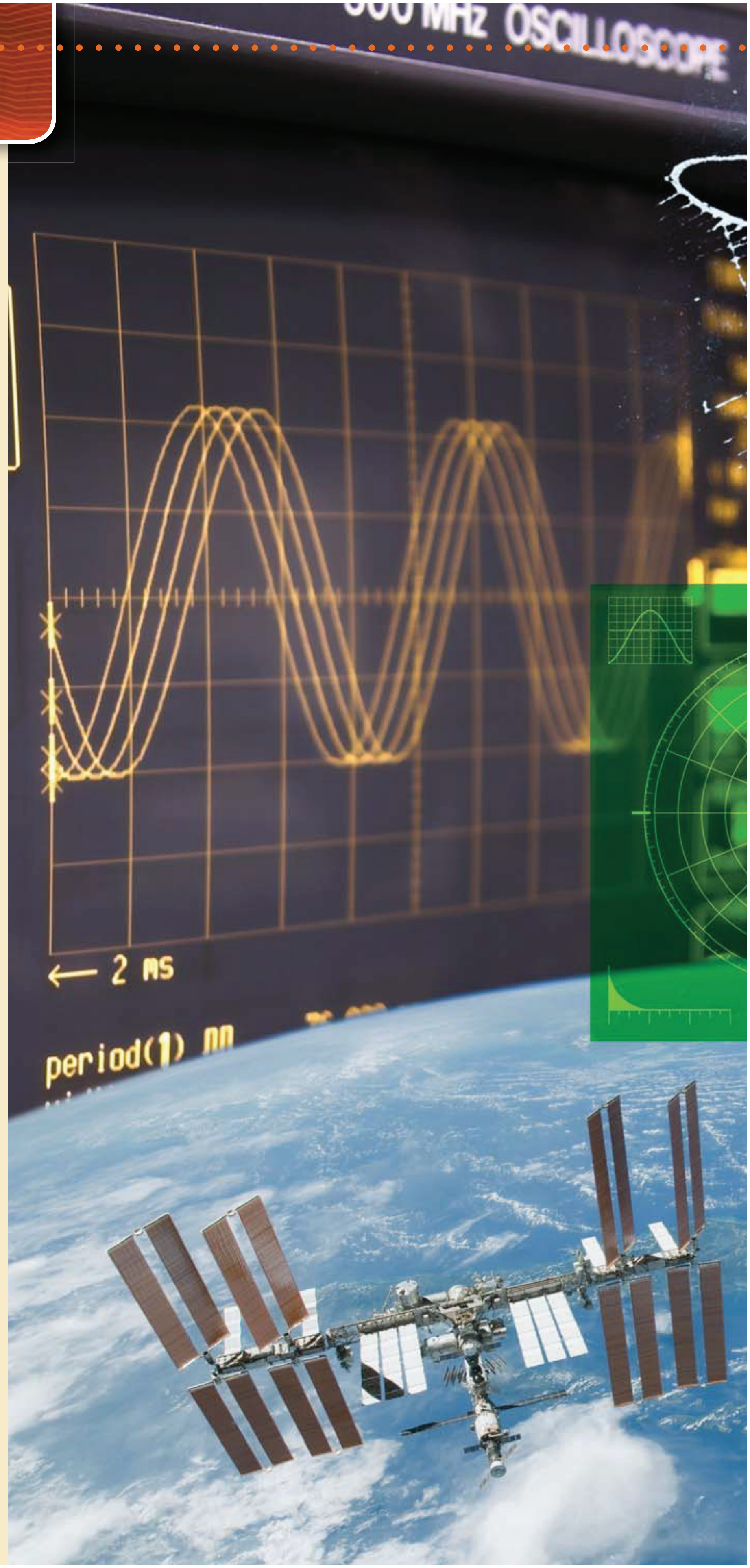
Unit 2

Trigonometry

Trigonometry is used extensively in our daily lives. For example, will you listen to music today? Most songs are recorded digitally and are compressed into MP3 format. These processes all involve trigonometry.

Your phone may have a built-in Global Positioning System (GPS) that uses trigonometry to tell where you are on Earth's surface. GPS satellites send a signal to receivers such as the one in your phone. The signal from each satellite can be represented using trigonometric functions. The receiver uses these signals to determine the location of the satellite and then uses trigonometry to calculate your position.



Looking Ahead

In this unit, you will solve problems involving…

- angle measures and the unit circle
- trigonometric functions and their graphs
- the proofs of trigonometric identities
- the solutions of trigonometric equations

Unit 2 Project Applications of Trigonometry

In this project, you will explore angle measurement, trigonometric equations, and trigonometric functions, and you will explore how they relate to past and present applications.

In Chapter 4, you will research the history of units of angle measure such as radians. In Chapter 5, you will gather information about the application of periodic functions to the field of communications. Finally, in Chapter 6, you will explore the use of trigonometric identities in Mach numbers.

At the end of the unit, you will choose at least one of the following options:
- Research the history, usage, and relationship of types of units for angle measure.
- Examine an application of periodic functions in electronic communications and investigate why it is an appropriate model.
- Apply the skills you have learned about trigonometric identities to supersonic travel.
- Explore the science of forensics through its applications of trigonometry.

CHAPTER 4

Trigonometry and the Unit Circle

Have you ever wondered about the repeating patterns that occur around us? Repeating patterns occur in sound, light, tides, time, and molecular motion. To analyse these repeating, cyclical patterns, you need to move from using ratios in triangles to using circular functions to approach trigonometry.

In this chapter, you will learn how to model and solve trigonometric problems using the unit circle and circular functions of radian measures.

Did You Know?

The flower in the photograph is called the Trigonometry daffodil. Why do you think this name was chosen?

Key Terms

radian
coterminal angles
general form
unit circle
cosecant
secant
cotangent
trigonometric equation

Career Link

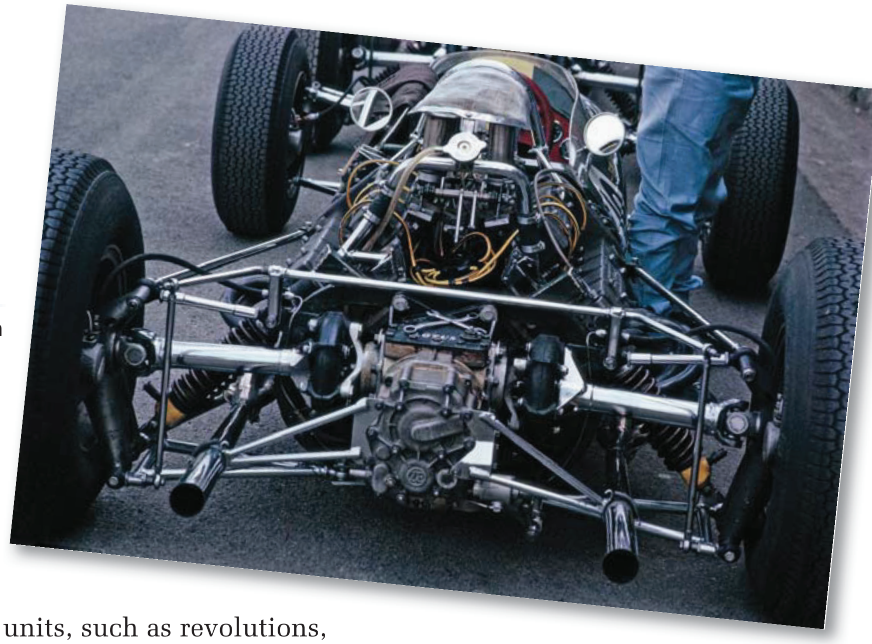
Engineers, police investigators, and legal experts all play key roles following a serious collision. Investigating and analysing a motor vehicle collision can provide valuable evidence for police and insurance reports. You can be trained in this fascinating and important field at police schools, engineering departments, and technical institutes.

Web Link

To learn more about accident reconstruction and training to become a forensic analysis investigator, go to www.mcgrawhill.ca/school/learningcentres and follow the links.

4.1

Angles and Angle Measure



Focus on...

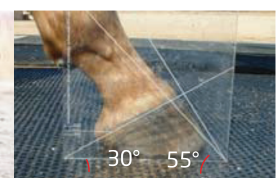
- sketching angles in standard position measured in degrees and radians
- converting angles in degree measure to radian measure and vice versa
- determining the measures of angles that are coterminal with a given angle
- solving problems involving arc lengths, central angles, and the radius in a circle

Angles can be measured using different units, such as revolutions, degrees, radians, and gradians. Which of these units are you familiar with? Check how many of these units are on your calculator.

Angles are everywhere and can be found in unexpected places. How many different angles can you see in the structure of the racing car?

Did You Know?

Sound (undamaged) hooves of all horses share certain angle aspects determined by anatomy and the laws of physics. A front hoof normally exhibits a 30° hairline and a 49° toe angle, while a hind hoof has a 30° hairline and a 55° toe angle.



Investigate Angle Measure

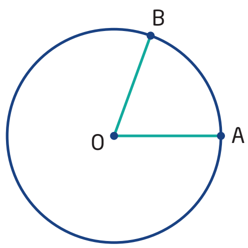
Materials

- masking tape
- sidewalk chalk
- string
- measuring tape

Work in small groups.

1. Mark the centre of a circle on the floor with sidewalk chalk. Then, using a piece of string greater than 1 m long for the radius, outline the circle with chalk or pieces of masking tape.
2. Label the centre of the circle O. Choose any point A on the circumference of the circle. OA is the radius of the circle. Have one member of your group walk heel-to-toe along the radius, counting foot lengths. Then, have the same person count the same number of foot lengths moving counterclockwise from A along the circumference. Label the endpoint B. Use tape to make the radii AO and BO. Have another member of the group confirm that the radius AO is the same length as arc AB.

3. Determine, by walking round the circle from B, approximately how many times the length of the radius fits onto the circumference.



Reflect and Respond

4. Use your knowledge of circumference to show that your answer in step 3 is reasonable.
5. Is $\angle$AOB in step 3 greater than, equal to, or less than 60°? Discuss this with your group.
6. Determine the degree measure of $\angle$AOB, to the nearest tenth of a degree.
7. Compare your results with those of other groups. Does the central angle AOB maintain its size if you use a larger circle? a smaller circle?

Link the Ideas

In the investigation, you encountered several key points associated with angle measure.

By convention, angles measured in a counterclockwise direction are said to be positive. Those measured in a clockwise direction are negative.

The angle AOB that you created measures 1 **radian**.

One full rotation is 360° or 2π radians.

One half rotation is 180° or π radians.

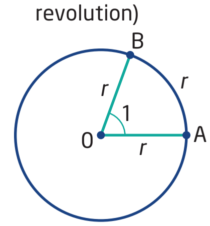
One quarter rotation is 90° or $\frac{\pi}{2}$ radians.

One eighth rotation is 45° or $\frac{\pi}{4}$ radians.

Many mathematicians omit units for radian measures. For example, $\frac{2\pi}{3}$ radians may be written as $\frac{2\pi}{3}$. Angle measures without units are considered to be in radians.

radian

- one radian is the measure of the central angle subtended in a circle by an arc equal in length to the radius of the circle
- $2\pi = 360°$
  $= 1$ full rotation (or revolution)



Example 1

Convert Between Degree and Radian Measure

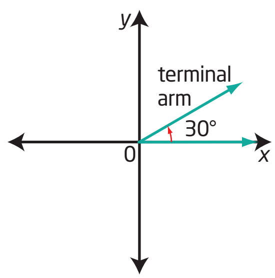
Draw each angle in standard position. Change each degree measure to radian measure and each radian measure to degree measure. Give answers as both exact and approximate measures (if necessary) to the nearest hundredth of a unit.

a) $30°$ **b)** $-120°$

c) $\frac{5\pi}{4}$ **d)** 2.57

Solution

a)



An angle in standard position has its centre at the origin and its initial arm along the positive *x*-axis.

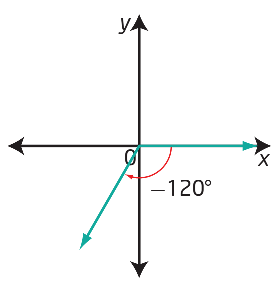
In which direction are positive angles measured?

Unitary Method

$$\begin{aligned} 360° &= 2\pi \\ 1° &= \frac{2\pi}{360} \\ &= \frac{\pi}{180} \\ 30° &= 30\left(\frac{\pi}{180}\right) \\ &= \frac{\pi}{6} \\ &\approx 0.52 \end{aligned}$$

$\frac{\pi}{6}$ is an exact value.

b)



Why is the angle drawn using a clockwise rotation?

Proportion Method

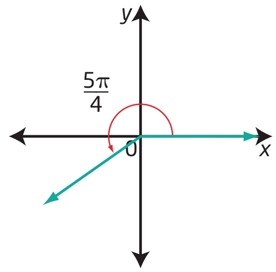
$$\begin{aligned} 180° &= \pi \\ \frac{-120°}{180°} &= \frac{x}{\pi} \\ x &= \frac{-120\pi}{180} \\ &= -\frac{2\pi}{3} \\ &\approx -2.09 \end{aligned}$$

So, $-120°$ is equivalent to $-\frac{2\pi}{3}$ or approximately -2.09.

c) π is $\frac{1}{2}$ rotation.

$\frac{\pi}{4}$ is $\frac{1}{8}$ rotation.

So $\frac{5\pi}{4}$ terminates in the third quadrant.



Unit Analysis

$$\frac{5\pi}{4} = \left(\frac{5\pi}{4}\right)\left(\frac{180°}{\pi}\right)$$
$$= \frac{5(180°)}{4}$$
$$= 225°$$

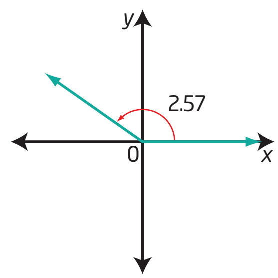
$\frac{5\pi}{4}$ is equivalent to 225°.

Why does $\left(\frac{180°}{\pi}\right)$ have value 1?

d) π (approximately 3.14) is $\frac{1}{2}$ rotation.

$\frac{\pi}{2}$ (approximately 1.57) is $\frac{1}{4}$ rotation.

2.57 is between 1.57 and 3.14,
so it terminates in the second quadrant.



Unitary Method

$$\pi = 180°$$
$$1 = \frac{180°}{\pi}$$

$$2.57$$
$$= 2.57\left(\frac{180°}{\pi}\right)$$
$$= \frac{462.6°}{\pi}$$
$$\approx 147.25°$$

Proportion Method

$$\frac{x}{2.57} = \frac{180°}{\pi}$$
$$x = 2.57\left(\frac{180°}{\pi}\right)$$
$$x = \frac{462.6°}{\pi}$$
$$x \approx 147.25°$$

Unit Analysis

$$2.57$$
$$= 2.57\left(\frac{180°}{\pi}\right)$$
$$= \frac{462.6°}{\pi}$$
$$\approx 147.25°$$

2.57 is equivalent to $\frac{462.6°}{\pi}$ or approximately 147.25°.

Your Turn

Draw each angle in standard position. Change each degree measure to radians and each radian measure to degrees. Give answers as both exact and approximate measures (if necessary) to the nearest hundredth of a unit.

a) $-270°$

b) $150°$

c) $\frac{7\pi}{6}$

d) -1.2

Did You Know?

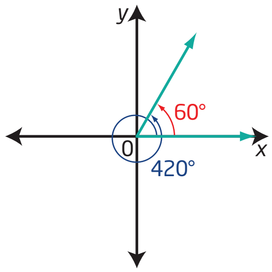
Most scientific and graphing calculators can calculate using angle measures in both degrees and radians. Find out how to change the mode on your calculator.

Coterminal Angles

When you sketch an angle of 60° and an angle of 420° in standard position, the terminal arms coincide. These are **coterminal angles**.

coterminal angles

- angles in standard position with the same terminal arms
- may be measured in degrees or radians
- $\frac{\pi}{4}$ and $\frac{9\pi}{4}$ are coterminal angles, as are 40° and −320°



Example 2

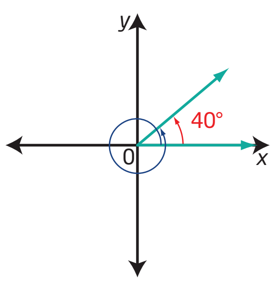
Identify Coterminal Angles

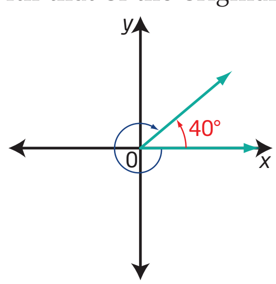
Determine one positive and one negative angle measure that is coterminal with each angle. In which quadrant does the terminal arm lie?

a) 40° **b)** −430° **c)** $\frac{8\pi}{3}$

Solution

a) The terminal arm is in quadrant I.
To locate coterminal angles, begin on the terminal arm of the given angle and rotate in a positive or negative direction until the new terminal arm coincides with that of the original angle.

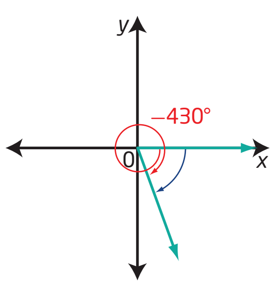


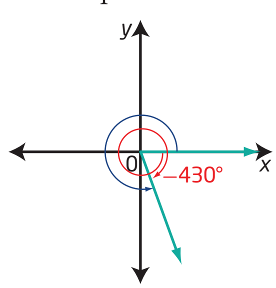


$40° + 360° = 400°$ $40° + (-360°) = -320°$
Two angles coterminal with 40° are 400° and −320°.

What other answers are possible?

b) The terminal arm of −430° is in quadrant IV.

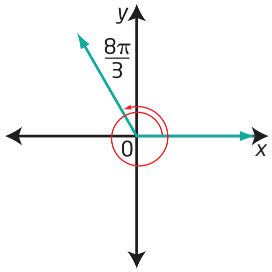




$-430° + 360° = -70°$ $-430° + 720° = 290°$
Two angles coterminal with −430° are 290° and −70°.

The reference angle is 70°.

c)



$\frac{8\pi}{3} = \frac{6\pi}{3} + \frac{2\pi}{3}$

So, the angle is one full rotation (2π) plus $\frac{2\pi}{3}$.

The terminal arm is in quadrant II.

There are 2π or $\frac{6\pi}{3}$ in one full rotation.

Counterclockwise one full rotation: $\frac{8\pi}{3} + \frac{6\pi}{3} = \frac{14\pi}{3}$

Clockwise one full rotation: $\frac{8\pi}{3} - \frac{6\pi}{3} = \frac{2\pi}{3}$

Clockwise two full rotations: $\frac{8\pi}{3} - \frac{12\pi}{3} = -\frac{4\pi}{3}$

Two angles coterminal with $\frac{8\pi}{3}$ are $\frac{2\pi}{3}$ and $-\frac{4\pi}{3}$.

Your Turn

For each angle in standard position, determine one positive and one negative angle measure that is coterminal with it.

a) 270° **b)** $-\frac{5\pi}{4}$ **c)** 740°

Coterminal Angles in General Form

By adding or subtracting multiples of one full rotation, you can write an infinite number of angles that are coterminal with any given angle.

For example, some angles that are coterminal with 40° are

$40° + (360°)(1) = 400°$ $\quad$ $40° - (360°)(1) = -320°$

$40° + (360°)(2) = 760°$ $\quad$ $40° - (360°)(2) = -680°$

In general, the angles coterminal with 40° are $40° \pm (360°)n$, where n is any natural number.

Some angles coterminal with $\frac{2\pi}{3}$ are

$$\frac{2\pi}{3} + 2\pi(1) = \frac{2\pi}{3} + \frac{6\pi}{3} = \frac{8\pi}{3} \qquad \frac{2\pi}{3} - 2\pi(1) = \frac{2\pi}{3} - \frac{6\pi}{3} = -\frac{4\pi}{3}$$

$$\frac{2\pi}{3} + 2\pi(2) = \frac{2\pi}{3} + \frac{12\pi}{3} = \frac{14\pi}{3} \qquad \frac{2\pi}{3} - 2\pi(2) = \frac{2\pi}{3} - \frac{12\pi}{3} = -\frac{10\pi}{3}$$

In general, the angles coterminal with $\frac{2\pi}{3}$ are $\frac{2\pi}{3} \pm 2\pi n$, where n is any natural number.

Any given angle has an infinite number of angles coterminal with it, since each time you make one full rotation from the terminal arm, you arrive back at the same terminal arm. Angles coterminal with any angle θ can be described using the expression

$\theta \pm (360°)n$ or $\theta \pm 2\pi n$,

where n is a natural number. This way of expressing an answer is called the **general form**.

general form

- an expression containing parameters that can be given specific values to generate any answer that satisfies the given information or situation
- represents all possible cases

Example 3

Express Coterminal Angles in General Form

a) Express the angles coterminal with 110° in general form. Identify the angles coterminal with 110° that satisfy the domain $-720° \le \theta < 720°$.

b) Express the angles coterminal with $\frac{8\pi}{3}$ in general form. Identify the angles coterminal with $\frac{8\pi}{3}$ in the domain $-4\pi \le \theta < 4\pi$.

Solution

a) Angles coterminal with 110° occur at $110° \pm (360°)n$, $n \in \mathrm{N}$.

Substitute values for n to determine these angles.

n	1	2	3
$110° - (360°)n$	$-250°$	$-610°$	$-970°$
$110° + (360°)n$	470°	830°	1190°

From the table, the values that satisfy the domain $-720° \le \theta < 720°$ are $-610°$, $-250°$, and 470°. These angles are coterminal.

b) $\frac{8\pi}{3} \pm 2\pi n$, $n \in \mathrm{N}$, represents all angles coterminal with $\frac{8\pi}{3}$.
Substitute values for n to determine these angles.

n	1	2	3	4
$\frac{8\pi}{3} - 2\pi n$	$\frac{2\pi}{3}$	$-\frac{4\pi}{3}$	$-\frac{10\pi}{3}$	$-\frac{16\pi}{3}$
$\frac{8\pi}{3} + 2\pi n$	$\frac{14\pi}{3}$	$\frac{20\pi}{3}$	$\frac{26\pi}{3}$	$\frac{32\pi}{3}$

The angles in the domain $-4\pi \le \theta < 4\pi$ that are coterminal are $-\frac{10\pi}{3}$, $-\frac{4\pi}{3}$, and $\frac{2\pi}{3}$.

Why is $-\frac{16\pi}{3}$ not an acceptable answer?

Your Turn

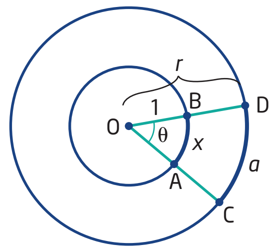
Write an expression for all possible angles coterminal with each given angle. Identify the angles that are coterminal that satisfy $-360° \le \theta < 360°$ or $-2\pi \le \theta < 2\pi$.

a) $-500°$ **b)** 650° **c)** $\frac{9\pi}{4}$

Arc Length of a Circle

All arcs that subtend a right angle $\left(\frac{\pi}{2}\right)$ have the same central angle, but they have different arc lengths depending on the radius of the circle. The arc length is proportional to the radius. This is true for any central angle and related arc length.

Consider two concentric circles with centre O. The radius of the smaller circle is 1, and the radius of the larger circle is r. A central angle of θ radians is subtended by arc AB on the smaller circle and arc CD on the larger one. You can write the following proportion, where x represents the arc length of the smaller circle and a is the arc length of the larger circle.



$$\frac{a}{x} = \frac{r}{1}$$
$$a = xr \qquad ①$$

Consider the circle with radius 1 and the sector with central angle θ. The ratio of the arc length to the circumference is equal to the ratio of the central angle to one full rotation.

$$\frac{x}{2\pi r} = \frac{\theta}{2\pi}$$

Why is $r = 1$?

$$x = \left(\frac{\theta}{2\pi}\right)2\pi(1)$$
$$x = \theta$$

Substitute $x = \theta$ in ①.

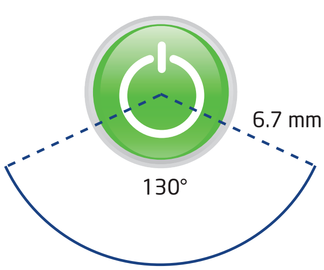
$a = \theta r$

This formula, $a = \theta r$, works for any circle, provided that θ is measured in radians and both a and r are measured in the same units.

Example 4

Determine Arc Length in a Circle

Rosemarie is taking a course in industrial engineering. For an assignment, she is designing the interface of a DVD player. In her plan, she includes a decorative arc below the on/off button. The arc has central angle 130° in a circle with radius 6.7 mm. Determine the length of the arc, to the nearest tenth of a millimetre.



Solution

Method 1: Convert to Radians and Use the Formula $a = \theta r$
Convert the measure of the central angle to radians before using the formula $a = \theta r$, where a is the arc length; θ is the central angle, in radians; and r is the length of the radius.

$$
\begin{aligned}
180° &= \pi \\
1° &= \frac{\pi}{180} \\
130° &= 130\left(\frac{\pi}{180}\right) \\
&= \frac{13\pi}{18}
\end{aligned}
$$

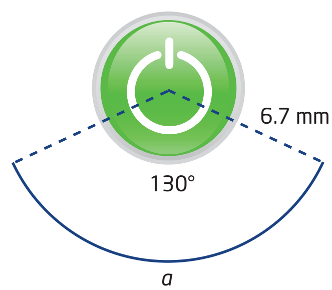
$$
\begin{aligned}
a &= \theta r \\
&= \left(\frac{13\pi}{18}\right)(6.7) \\
&= \frac{87.1\pi}{18} \\
&= 15.201...
\end{aligned}
$$

Why is it important to use exact values throughout the calculation and only convert to decimal fractions at the end?

The arc length is 15.2 mm, to the nearest tenth of a millimetre.

Method 2: Use a Proportion
Let a represent the arc length.

$$
\begin{aligned}
\frac{\text{arc length}}{\text{circumference}} &= \frac{\text{central angle}}{\text{full rotation}} \\
\frac{a}{2\pi(6.7)} &= \frac{130°}{360°} \\
a &= \frac{2\pi(6.7)130°}{360°} \\
&= 15.201...
\end{aligned}
$$



The arc length is 15.2 mm, to the nearest tenth of a millimetre.

Your Turn

If a represents the length of an arc of a circle with radius r, subtended by a central angle of θ, determine the missing quantity. Give your answers to the nearest tenth of a unit.

a) $r = 8.7$ cm, $\theta = 75°$, $a = ■$ cm
b) $r = ■$ mm, $\theta = 1.8$, $a = 4.7$ mm
c) $r = 5$ m, $a = 13$ m, $\theta = ■$

Key Ideas

- Angles can be measured using different units, including degrees and radians.
- An angle measured in one unit can be converted to the other unit using the relationships 1 full rotation $= 360° = 2\pi$.
- An angle in standard position has its vertex at the origin and its initial arm along the positive x-axis.
- Angles that are coterminal have the same initial arm and the same terminal arm.
- An angle θ has an infinite number of angles that are coterminal expressed by $\theta \pm (360°)n$, $n \in \mathrm{N}$, in degrees, or $\theta \pm 2\pi n$, $n \in \mathrm{N}$, in radians.
- The formula $a = \theta r$, where a is the arc length; θ is the central angle, in radians; and r is the length of the radius, can be used to determine any of the variables given the other two, as long as a and r are in the same units.

Check Your Understanding

Practise

1. For each angle, indicate whether the direction of rotation is clockwise or counterclockwise.

a) -4π **b)** $750°$
c) $-38.7°$ **d)** 1

2. Convert each degree measure to radians. Write your answers as exact values. Sketch the angle and label it in degrees and in radians.

a) $30°$ **b)** $45°$
c) $-330°$ **d)** $520°$
e) $90°$ **f)** $21°$

3. Convert each degree measure to radians. Express your answers as exact values and as approximate measures, to the nearest hundredth of a radian.

a) $60°$ **b)** $150°$
c) $-270°$ **d)** $72°$
e) $-14.8°$ **f)** $540°$

4. Convert each radian measure to degrees. Express your answers as exact values and as approximate measures, to the nearest tenth of a degree, if necessary.

a) $\frac{\pi}{6}$ **b)** $\frac{2\pi}{3}$
c) $-\frac{3\pi}{8}$ **d)** $-\frac{5\pi}{2}$
e) 1 **f)** 2.75

5. Convert each radian measure to degrees. Express your answers as exact values and as approximate measures, to the nearest thousandth.

a) $\frac{2\pi}{7}$ **b)** $\frac{7\pi}{13}$
c) $\frac{2}{3}$ **d)** 3.66
e) -6.14 **f)** -20

6. Sketch each angle in standard position. In which quadrant does each angle terminate?

a) 1 **b)** $-225°$
c) $\frac{17\pi}{6}$ **d)** $650°$
e) $-\frac{2\pi}{3}$ **f)** $-42°$

7. Determine one positive and one negative angle coterminal with each angle.

a) $72°$
b) $\frac{3\pi}{4}$
c) $-120°$
d) $\frac{11\pi}{2}$
e) $-205°$
f) 7.8

8. Determine whether the angles in each pair are coterminal. For one pair of angles, explain how you know.

a) $\frac{5\pi}{6}, \frac{17\pi}{6}$
b) $\frac{5\pi}{2}, -\frac{9\pi}{2}$
c) $410°, -410°$
d) $227°, -493°$

9. Write an expression for all of the angles coterminal with each angle. Indicate what your variable represents.

a) $135°$
b) $-\frac{\pi}{2}$
c) $-200°$
d) 10

10. Draw and label an angle in standard position with negative measure. Then, determine an angle with positive measure that is coterminal with your original angle. Show how to use a general expression for coterminal angles to find the second angle.

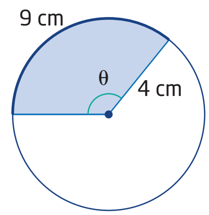
11. For each angle, determine all angles that are coterminal in the given domain.

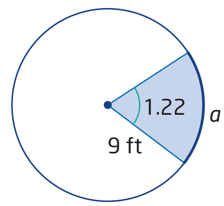
a) $65°, 0° \leq \theta < 720°$
b) $-40°, -180° \leq \theta < 360°$
c) $-40°, -720° \leq \theta < 720°$
d) $\frac{3\pi}{4}, -2\pi \leq \theta < 2\pi$
e) $-\frac{11\pi}{6}, -4\pi \leq \theta < 4\pi$
f) $\frac{7\pi}{3}, -2\pi \leq \theta < 4\pi$
g) $2.4, -2\pi \leq \theta < 2\pi$
h) $-7.2, -4\pi \leq \theta < 2\pi$

12. Determine the arc length subtended by each central angle. Give answers to the nearest hundredth of a unit.

a) radius 9.5 cm, central angle 1.4
b) radius 1.37 m, central angle 3.5
c) radius 7 cm, central angle 130°
d) radius 6.25 in., central angle 282°

13. Use the information in each diagram to determine the value of the variable. Give your answers to the nearest hundredth of a unit.

a)


b)


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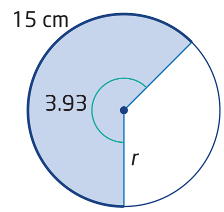

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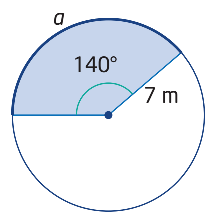


Apply

14. A rotating water sprinkler makes one revolution every 15 s. The water reaches a distance of 5 m from the sprinkler.

a) What is the arc length of the sector watered when the sprinkler rotates through $\frac{5\pi}{3}$? Give your answer as both an exact value and an approximate measure, to the nearest hundredth.
b) Show how you could find the area of the sector watered in part a).
c) What angle does the sprinkler rotate through in 2 min? Express your answer in radians and degrees.

15. Angular velocity describes the rate of change in a central angle over time. For example, the change could be expressed in revolutions per minute (rpm), radians per second, degrees per hour, and so on. All that is required is an angle measurement expressed over a unit of time.

a) Earth makes one revolution every 24 h. Express the angular velocity of Earth in three other ways.

b) An electric motor rotates at 1000 rpm. What is this angular velocity expressed in radians per second?

c) A bicycle wheel completes 10 revolutions every 4 s. Express this angular velocity in degrees per minute.

16. Skytrek Adventure Park in Revelstoke, British Columbia, has a sky swing. Can you imagine a 170-ft flight that takes riders through a scary pendulum swing? At one point you are soaring less than 10 ft from the ground at speeds exceeding 60 mph.

a) The length of the cable is 72 ft and you travel on an arc of length 170 ft on one particular swing. What is the measure of the central angle? Give your answer in radians, to the nearest hundredth.

b) What is the measure of the central angle from part a), to the nearest tenth of a degree?

17. Copy and complete the table by converting each angle measure to its equivalent in the other systems. Round your answers to the nearest tenth where necessary.

	Revolutions	Degrees	Radians
a)	1 rev		
b)		270°	
c)			$\frac{5\pi}{6}$
d)			−1.7
e)		−40°	
f)	0.7 rev		
g)	−3.25 rev		
h)		460°	
i)			$-\frac{3\pi}{8}$

18. Joran and Jasmine are discussing expressions for the general form of coterminal angles of 78°. Joran claims the answer must be expressed as $78° + (360°)n$, $n \in \mathrm{I}$. Jasmine indicates that although Joran's expression is correct, another answer is possible. She prefers $78° \pm k(360°)$, $k \in \mathrm{N}$, where N represents positive integers. Who is correct? Why?

19. The gradian (grad) is another unit of angle measure. It is defined as $\frac{1}{400}$ of a revolution, so one full rotation contains 400 grads.

a) Determine the number of gradians in 50°.

b) Describe a process for converting from degree measure to gradians and vice versa.

c) Identify a possible reason that the gradian was created.

Did You Know?

Gradians originated in France in the 1800s. They are still used in some engineering work.

20. Yellowknife, Northwest Territories, and Crowsnest Pass, Alberta, lie along the 114° W line of longitude. The latitude of Yellowknife is 62.45° N and the latitude of Crowsnest Pass is 49.63° N. Consider Earth to be a sphere with radius 6400 km.

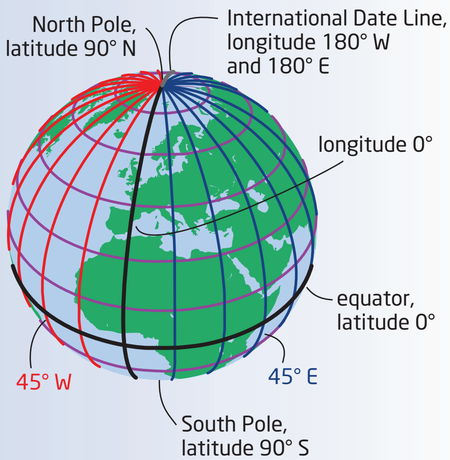
   a) Sketch the information given above using a circle. Label the centre of Earth, its radius to the equator, and the locations of Yellowknife and Crowsnest Pass.

   b) Determine the distance between Yellowknife and Crowsnest Pass. Give your answer to the nearest hundredth of a kilometre.

   c) Choose a town or city either where you live or nearby. Determine the latitude and longitude of this location. Find another town or city with the same longitude. What is the distance between the two places?

Did You Know?

Lines of latitude and longitude locate places on Earth. Lines of latitude are parallel to the equator and are labelled from 0° at the equator to 90° at the North Pole. Lines of longitude converge at the poles and are widest apart at the equator. 0° passes through Greenwich, England, and the lines are numbered up to 180° E and 180° W, meeting at the International Date Line.



21. Sam Whittingham from Quadra Island, British Columbia, holds five 2009 world human-powered speed records on his recumbent bicycle. In the 200-m flying start, he achieved a speed of 133.284 km/h.

   a) Express the speed in metres per minute.

   b) The diameter of his bicycle wheel is 60 cm. Through how many radians per minute must the wheels turn to achieve his world record in the 200-m flying start?

22. A water wheel with diameter 3 m is used to measure the approximate speed of the water in a river. If the angular velocity of the wheel is 15 rpm, what is the speed of the river, in kilometres per hour?

23. Earth is approximately 93 000 000 mi from the sun. It revolves around the sun, in an almost circular orbit, in about 365 days. Calculate the linear speed, in miles per hour, of Earth in its orbit. Give your answer to the nearest hundredth.

Extend

24. Refer to the Did You Know? below.

   a) With a partner, show how to convert 69.375° to 69° 22′ 30″.

   b) Change the following angles into degrees-minutes-seconds.

   i) 40.875° **ii)** 100.126°
   iii) 14.565° **iv)** 80.385°

Did You Know?

You have expressed degree measures as decimal numbers, for example, 69.375°. Another way subdivides 1° into 60 parts called minutes. Each minute can be subdivided into 60 parts called seconds. Then, an angle such as 69.375° can be written as 69° 22 min 30 s or 69° 22′ 30″.

25. **a)** Reverse the process of question 24 and show how to convert 69° 22′ 30″ to 69.375°. Hint: Convert 30″ into a decimal fraction part of a minute. Combine this part of a minute with the 22′ and then convert the minutes to part of a degree.

b) Change each angle measure into degrees, rounded to the nearest thousandth.

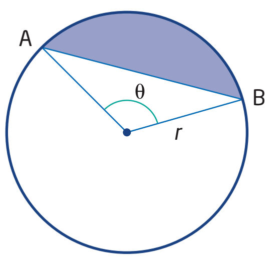
i) 45° 30′ 30″

ii) 72° 15′ 45″

iii) 105° 40′ 15″

iv) 28° 10′

26. A segment of a circle is the region between a chord and the arc subtended by that chord. Consider chord AB subtended by central angle θ in a circle with radius *r*. Derive a formula using *r* and θ for the area of the segment subtended by θ.



27. The hour hand of an analog clock moves in proportion to the movement of the minute hand. This means that at 4:05, the hour hand will have moved beyond the 4 by $\frac{5}{60}$ of the distance it would move in an hour.

a) What is the measure of the obtuse angle between the hands of a clock at 4:00? Give your answer in degrees.

b) What is the measure, in degrees, of the acute angle between the hands of a clock at 4:10?

c) At certain times, the hands of a clock are at right angles to each other. What are two of these times?

d) At how many different times does the angle between the hands of a clock measure 90° between 4:00 and 5:00?

e) Does one of the times occur before, at, or shortly after 4:05? Explain.

Create Connections

C1 Draw a diagram and use it to help explain whether 6 radians is greater than, equal to, or less than 360°.

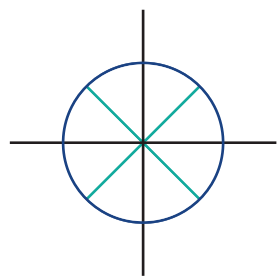
C2 In mathematics, angle measures are commonly defined in degrees or radians. Describe the difference between 1° and 1 radian. Use drawings to support your answer.

C3 The following angles are in standard position. What is the measure of the reference angle for each? Write an expression for all coterminal angles associated with each given angle.

a) 860°

b) −7 (give the reference angle to the nearest hundredth)

C4 **a)** Make a circle diagram similar to the one shown. On the outside of the circle, label all multiples of 45° in the domain $0° \leq \theta < 360°$. Show the radian equivalent as an exact value inside the circle.



b) Make another circle diagram. This time, mark and label all the multiples of 30° in the domain $0° \leq \theta < 360°$. Again, show the degree values outside the circle and the exact radian equivalents inside the circle.

C5 A line passes through the point (3, 0). Find the equation of the line if the angle formed between the line and the positive *x*-axis is

a) $\frac{\pi}{2}$ **b)** 45°

4.2

The Unit Circle

Focus on…

- developing and applying the equation of the unit circle
- generalizing the equation of a circle with centre (0, 0) and radius r
- using symmetry and patterns to locate the coordinates of points on the unit circle

A gauge is a measuring tool that is used in many different situations. There are two basic types of gauges—radial (circular) and linear. What gauges can you think of that are linear? What gauges are you familiar with that are circular? How are linear and circular gauges similar, and how do they differ?

Have you ever wondered why some phenomena, such as tides and hours of daylight, are so predictable? It is because they have repetitive or cyclical patterns. Why is sin 30° the same as sin 150°? Why is cos 60° = sin 150°? How do the coordinates of a point on a circle of radius 1 unit change every quarter-rotation?

Investigate Circular Number Lines

Materials

- paper
- scissors
- tape
- can or other cylinder
- straight edge
- compass

1. Select a can or other cylinder. Cut a strip of paper about 1.5 cm wide and the same length as the circumference of the cylinder.

2. Create a number line by drawing a line along the centre of the strip. Label the left end of the line 0 and the right end 2π. According to this labelling, how long is the number line?

3. Divide the number line into eight equal subdivisions. What value would you use to label the point midway between 0 and 2π? What value would you use to label halfway between 0 and the middle of the number line? Continue until all seven points that subdivide the number line are labelled. Write all values in terms of π. Express fractional values in lowest terms.

4. Tape the number line around the bottom of the can, so that the labels read in a counterclockwise direction.

5. Use the can to draw a circle on a sheet of paper. Locate the centre of the circle and label it O. Draw coordinate axes through O that extend beyond the circle. Place the can over the circle diagram so that the zero of the number line lies above where the circle intersects the positive x-axis.

6. Mark the coordinates of all points where the circle crosses the axes on your diagram. Label these points as $P(\theta) = (x, y)$, where $P(\theta)$ represents a point on the circle that has a central angle θ in standard position. For example, label the point where the circle crosses the positive y-axis as $P\left(\frac{\pi}{2}\right) = (0, 1)$.
7. Now, create a second number line. Label the ends as 0 and 2π. Divide this number line into 12 equal segments. Label divisions in terms of π. Express fractional values in lowest terms.

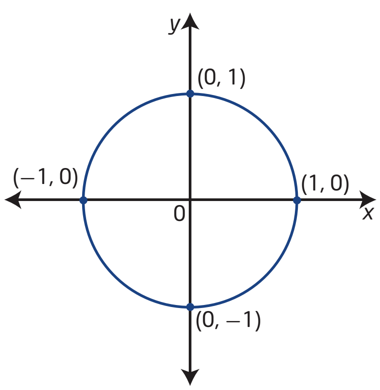
Reflect and Respond

8. Since each number line shows the circumference of the can and the circle to be 2π units, what assumption is being made about the length of the radius?
9. **a)** Two students indicate that the points in step 6 are simply multiples of $\frac{\pi}{2}$. Do you agree? Explain.
   b) In fact, they argue that the values on the original number line are all multiples of $\frac{\pi}{4}$. Is this true? Explain.
10. Show how to determine the coordinates for $P\left(\frac{\pi}{4}\right)$. Hint: Use your knowledge of the ratios of the side lengths of a 45°-45°-90° triangle. Mark the coordinates for all the points on the circle that are midway between the axes. What is the only difference in the coordinates for these four points? What negative values for θ would generate the same points on the circle midway between the axes?

Link the Ideas

Unit Circle

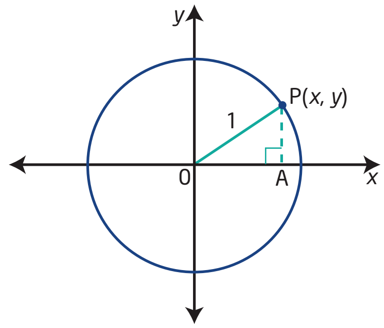
The circle you drew in the investigation is a **unit circle**.



unit circle
- a circle with radius 1 unit
- a circle of radius 1 unit with centre at the origin on the Cartesian plane is known as *the* unit circle

You can find the equation of the unit circle using the Pythagorean theorem.

Consider a point P on the unit circle. Let P have coordinates (x, y). Draw right triangle OPA as shown.



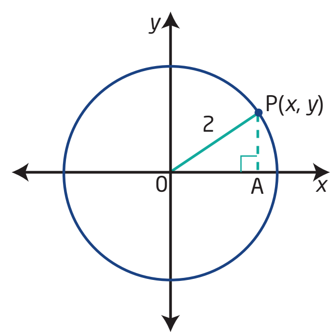
$OP = 1$	The radius of the unit circle is 1.
$PA = \lvert y \rvert$	The absolute value of the *y*-coordinate represents the distance from a point to the *x*-axis. Why is this true?
$OA = \lvert x \rvert$	
$(OP)^2 = (OA)^2 + (PA)^2$	Pythagorean theorem
$1^2 = \lvert x \rvert^2 + \lvert y \rvert^2$	How would the equation for a circle with centre O(0, 0) differ if the radius were *r* rather than 1?
$1 = x^2 + y^2$	

The equation of the unit circle is $x^2 + y^2 = 1$.

Example 1

Equation of a Circle Centred at the Origin

Determine the equation of the circle with centre at the origin and radius 2.



Solution

Choose a point, P, on the circle with coordinates (x, y).

The radius of the circle is 2, and a vertical line from the *y*-coordinate to the *x*-axis forms a right angle with the axis. This means you can use the Pythagorean theorem.

$\lvert x \rvert^2 + \lvert y \rvert^2 = 2^2$
$x^2 + y^2 = 4$

Since this is true for every point P on the circle, the equation of the circle is $x^2 + y^2 = 4$.

Your Turn

Determine the equation of a circle with centre at the origin and radius 6.

Example 2

Determine Coordinates for Points of the Unit Circle

Determine the coordinates for all points on the unit circle that satisfy the conditions given. Draw a diagram in each case.

a) the x-coordinate is $\frac{2}{3}$

b) the y-coordinate is $-\frac{1}{\sqrt{2}}$ and the point is in quadrant III

Solution

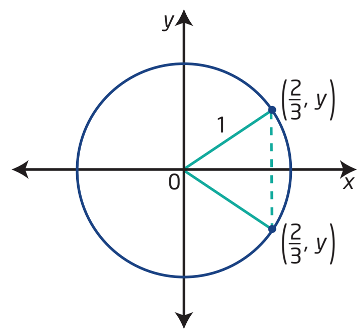
a) Coordinates on the unit circle satisfy the equation $x^2 + y^2 = 1$.

$$\left(\frac{2}{3}\right)^2 + y^2 = 1$$

Since x is positive, which quadrants could the points be in?

$$\frac{4}{9} + y^2 = 1$$
$$y^2 = \frac{5}{9}$$
$$y = \pm\frac{\sqrt{5}}{3}$$

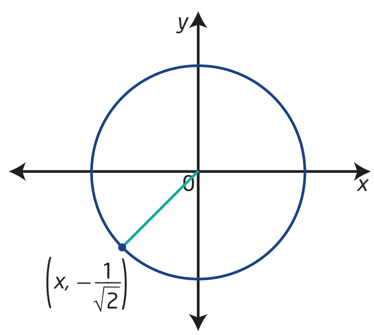
Why are there two answers?



Two points satisfy the given conditions: $\left(\frac{2}{3}, \frac{\sqrt{5}}{3}\right)$ in quadrant I and $\left(\frac{2}{3}, -\frac{\sqrt{5}}{3}\right)$ in quadrant IV.

b) $y = -\frac{1}{\sqrt{2}}$

y is negative in quadrants III and IV. But the point is in quadrant III, so x is also negative.



$$x^2 + y^2 = 1$$
$$x^2 + \left(-\frac{1}{\sqrt{2}}\right)^2 = 1$$
$$x^2 + \frac{1}{2} = 1$$
$$x^2 = \frac{1}{2}$$
$$x = -\frac{1}{\sqrt{2}}$$

Why is there only one answer?

The point is $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, or $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

Your Turn

Determine the missing coordinate(s) for all points on the unit circle satisfying the given conditions. Draw a diagram and tell which quadrant(s) the points lie in.

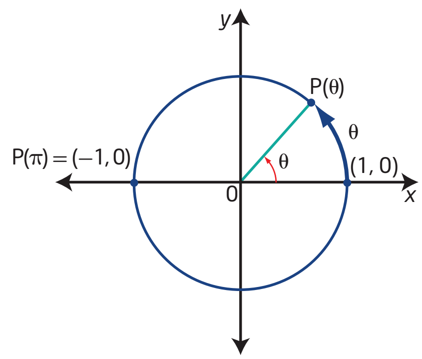
a) $\left(-\frac{5}{8}, y\right)$ **b)** $\left(x, \frac{5}{13}\right)$, where the point is in quadrant II

Relating Arc Length and Angle Measure in Radians

The formula $a = \theta r$, where a is the arc length; θ is the central angle, in radians; and r is the radius, applies to any circle, as long as a and r are measured in the same units. In the unit circle, the formula becomes $a = \theta(1)$ or $a = \theta$. This means that a central angle and its subtended arc on the unit circle have the same numerical value.

You can use the function $P(\theta) = (x, y)$ to link the arc length, θ, of a central angle in the unit circle to the coordinates, (x, y), of the point of intersection of the terminal arm and the unit circle.

If you join $P(\theta)$ to the origin, you create an angle θ in standard position. Now, θ radians is the central angle and the arc length is θ units.



Function P takes real-number values for the central angle or the arc length on the unit circle and matches them with specific points. For example, if $\theta = \pi$, the point is $(-1, 0)$. Thus, you can write $P(\pi) = (-1, 0)$.

Example 3

Multiples of $\frac{\pi}{3}$ on the Unit Circle

a) On a diagram of the unit circle, show the integral multiples of $\frac{\pi}{3}$ in the interval $0 \leq \theta \leq 2\pi$.

b) What are the coordinates for each point $P(\theta)$ in part a)?

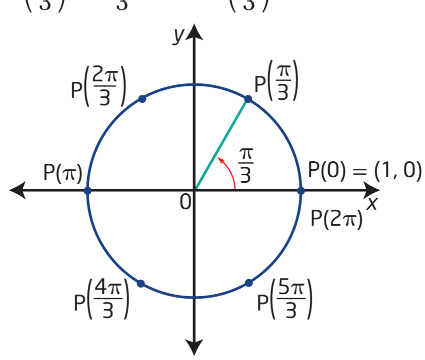
c) Identify any patterns you see in the coordinates of the points.

Solution

a) This is essentially a counting problem using $\frac{\pi}{3}$.

Multiples of $\frac{\pi}{3}$ in the interval $0 \leq \theta \leq 2\pi$ are

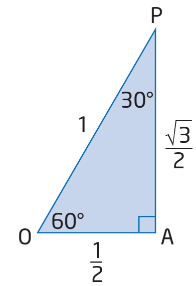
$0\left(\frac{\pi}{3}\right) = 0$, $1\left(\frac{\pi}{3}\right) = \frac{\pi}{3}$, $2\left(\frac{\pi}{3}\right) = \frac{2\pi}{3}$, $3\left(\frac{\pi}{3}\right) = \pi$, $4\left(\frac{\pi}{3}\right) = \frac{4\pi}{3}$, $5\left(\frac{\pi}{3}\right) = \frac{5\pi}{3}$, and $6\left(\frac{\pi}{3}\right) = 2\pi$.



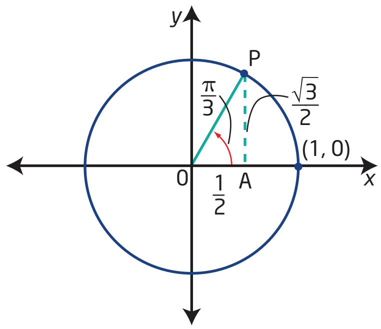
Why must you show only the multiples in one positive rotation in the unit circle?

b) Recall that a 30°-60°-90° triangle has sides in the ratio $1:\sqrt{3}:2$ or $\frac{1}{2}:\frac{\sqrt{3}}{2}:1$.

Why is the 30°-60°-90° triangle used?

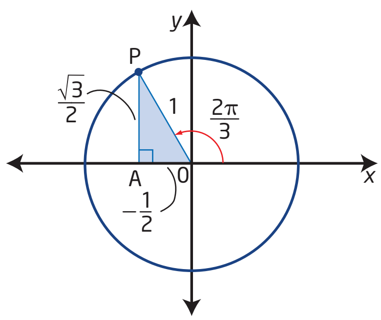


Place $\triangle$POA in the unit circle as shown.



Why are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ the coordinates of $P\left(\frac{\pi}{3}\right)$?

$\triangle$POA could be placed in the second quadrant with O at the origin and OA along the *x*-axis as shown. This gives $P\left(\frac{2\pi}{3}\right) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.



Why is the *x*-coordinate negative?

What transformation could be used to move $\triangle$POA from quadrant I to quadrant II?

Continue, placing $\triangle$POA in quadrants III and IV to find the coordinates of $P\left(\frac{4\pi}{3}\right)$ and $P\left(\frac{5\pi}{3}\right)$. Then, the coordinates of point P corresponding to angles that are multiples of $\frac{\pi}{3}$ are

$P(0) = P(2\pi) = (1, 0)$ $\quad$ $P(\pi) = (-1, 0)$ $\quad$ $P\left(\frac{\pi}{3}\right) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

$P\left(\frac{2\pi}{3}\right) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ $\quad$ $P\left(\frac{4\pi}{3}\right) = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ $\quad$ $P\left(\frac{5\pi}{3}\right) = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

c) Some patterns are:

- The points corresponding to angles that are multiples of $\frac{\pi}{3}$ that cannot be simplified, for example, $P\left(\frac{\pi}{3}\right)$, $P\left(\frac{2\pi}{3}\right)$, $P\left(\frac{4\pi}{3}\right)$, and $P\left(\frac{5\pi}{3}\right)$, have the same coordinates except for their signs.
- Any points where θ reduces to a multiple of π, for example, $P(0)$, $P\left(\frac{3\pi}{3}\right) = P(\pi)$, and $P\left(\frac{6\pi}{3}\right) = P(2\pi)$, fall on an axis.

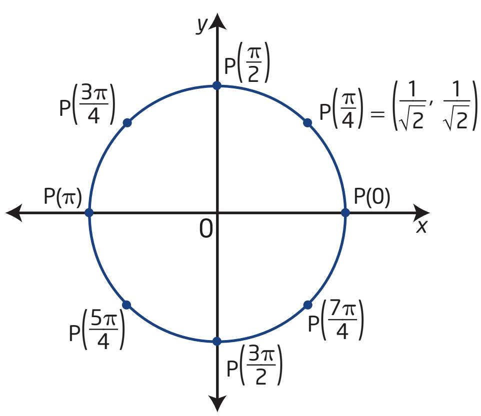
Your Turn

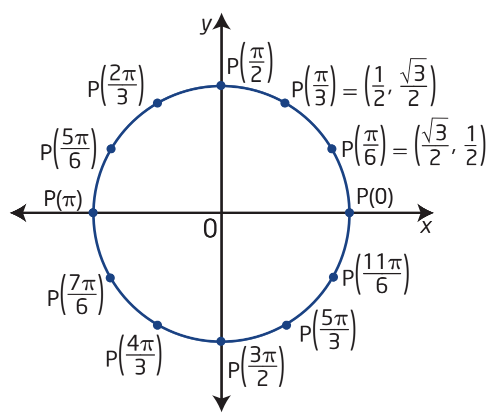
a) On a diagram of the unit circle, show all the integral multiples of $\frac{\pi}{6}$ in the interval $0 \leq \theta < 2\pi$.
b) Label the coordinates for each point $P(\theta)$ on your diagram.
c) Describe any patterns you see in the coordinates of the points.

Key Ideas

- The equation for the unit circle is $x^2 + y^2 = 1$. It can be used to determine whether a point is on the unit circle or to determine the value of one coordinate given the other. The equation for a circle with centre at (0, 0) and radius r is $x^2 + y^2 = r^2$.
- On the unit circle, the measure in radians of the central angle and the arc subtended by that central angle are numerically equivalent.
- Some of the points on the unit circle correspond to exact values of the special angles learned previously.
- You can use patterns to determine coordinates of points. For example, the numerical value of the coordinates of points on the unit circle change to their opposite sign every $\frac{1}{2}$ rotation.

  If $P(\theta) = (a, b)$ is in quadrant I, then both a and b are positive. $P(\theta + \pi)$ is in quadrant III. Its coordinates are $(-a, -b)$, where $a > 0$ and $b > 0$.





Check Your Understanding

Practise

1. Determine the equation of a circle with centre at the origin and radius

a) 4 units

b) 3 units

c) 12 units

d) 2.6 units

2. Is each point on the unit circle? How do you know?

a) $\left(-\frac{3}{4}, \frac{1}{4}\right)$

b) $\left(\frac{\sqrt{5}}{8}, \frac{7}{8}\right)$

c) $\left(-\frac{5}{13}, \frac{12}{13}\right)$

d) $\left(\frac{4}{5}, -\frac{3}{5}\right)$

e) $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

f) $\left(\frac{\sqrt{7}}{4}, \frac{3}{4}\right)$

3. Determine the missing coordinate(s) for all points on the unit circle satisfying the given conditions. Draw a diagram to support your answer.

a) $\left(\frac{1}{4}, y\right)$ in quadrant I

b) $\left(x, \frac{2}{3}\right)$ in quadrant II

c) $\left(-\frac{7}{8}, y\right)$ in quadrant III

d) $\left(x, -\frac{5}{7}\right)$ in quadrant IV

e) $\left(x, \frac{1}{3}\right)$, where $x < 0$

f) $\left(\frac{12}{13}, y\right)$, not in quadrant I

4. If P(θ) is the point at the intersection of the terminal arm of angle θ and the unit circle, determine the exact coordinates of each of the following.

a) $P(\pi)$ **b)** $P\left(-\frac{\pi}{2}\right)$

c) $P\left(\frac{\pi}{3}\right)$ **d)** $P\left(-\frac{\pi}{6}\right)$

e) $P\left(\frac{3\pi}{4}\right)$ **f)** $P\left(-\frac{7\pi}{4}\right)$

g) $P(4\pi)$ **h)** $P\left(\frac{5\pi}{2}\right)$

i) $P\left(\frac{5\pi}{6}\right)$ **j)** $P\left(-\frac{4\pi}{3}\right)$

5. Identify a measure for the central angle θ in the interval $0 \le \theta < 2\pi$ such that P(θ) is the given point.

a) $(0, -1)$ **b)** $(1, 0)$

c) $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ **d)** $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

e) $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ **f)** $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

g) $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ **h)** $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

i) $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ **j)** $(-1, 0)$

6. Determine one positive and one negative measure for θ if $P(\theta) = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

Apply

7. Draw a diagram of the unit circle.

a) Mark two points, P(θ) and P(θ + π), on your diagram. Use measurements to show that these points have the same coordinates except for their signs.

b) Choose a different quadrant for the original point, P(θ). Mark it and P(θ + π) on your diagram. Is the result from part a) still true?

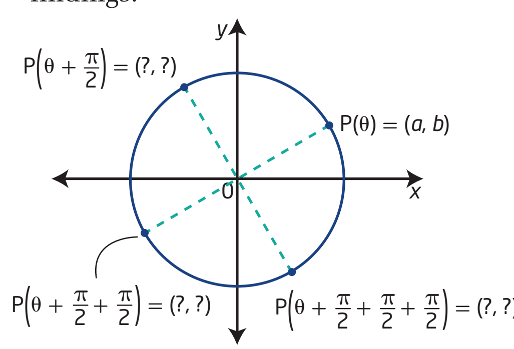
8. MINI LAB Determine the pattern in the coordinates of points that are $\frac{1}{4}$ rotation apart on the unit circle.

Step 1 Start with the points $P(0) = (1, 0)$, $P\left(\frac{\pi}{3}\right) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, and $P\left(\frac{5\pi}{3}\right) = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. Show these points on a diagram.

Step 2 Move $+\frac{1}{4}$ rotation from each point. Determine each new point and its coordinates. Show these points on your diagram from step 1.

Step 3 Move $-\frac{1}{4}$ rotation from each original point. Determine each new point and its coordinates. Mark these points on your diagram.

Step 4 How do the values of the *x*-coordinates and *y*-coordinates of points change with each quarter-rotation? Make a copy of the diagram and complete the coordinates to summarize your findings.



9. Use the diagram below to help answer these questions.

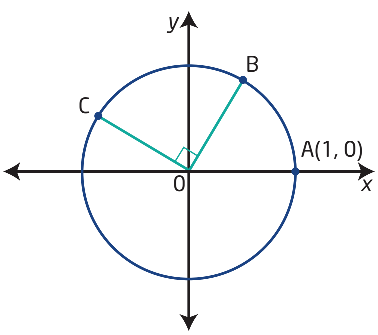
   a) What is the equation of this circle?

   b) If the coordinates of C are $\left(-\frac{2}{3}, \frac{\sqrt{5}}{3}\right)$, what are the coordinates of B?

   c) If the measure of $\overset{\frown}{AB}$ is θ, what is an expression for the measure of $\overset{\frown}{AC}$?
   Note: $\overset{\frown}{AB}$ means the arc length from A to B.

   d) Let $P(\theta) = B$. In which quadrant is $P\left(\theta - \frac{\pi}{2}\right)$?

   e) What are the maximum and minimum values for either the *x*-coordinates or *y*-coordinates of points on the unit circle?



10. Mya claims that every value of *x* between 0 and 1 can be used to find the coordinates of a point on the unit circle in quadrant I.

   a) Do you agree with Mya? Explain.

   b) Mya showed the following work to find the *y*-coordinate when $x = 0.807$.

   $y = 1 - (0.807)^2$
   $= 0.348\ 751$

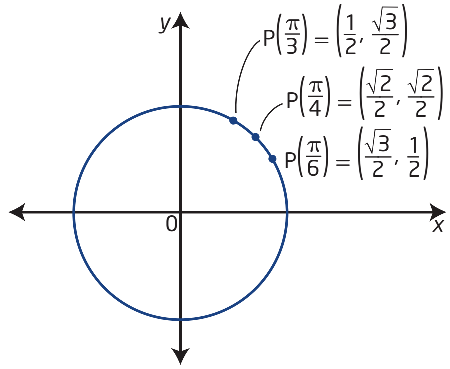
   The point on the unit circle is (0.807, 0.348 751).

   How can you check Mya's answer? Is she correct? If not, what is the correct answer?

   c) If $y = 0.2571$, determine *x* so the point is on the unit circle and in the first quadrant.

11. Wesley enjoys tricks and puzzles. One of his favourite tricks involves remembering the coordinates for $P\left(\frac{\pi}{3}\right)$, $P\left(\frac{\pi}{4}\right)$, and $P\left(\frac{\pi}{6}\right)$. He will not tell you his trick. However, you can discover it for yourself.

   a) Examine the coordinates shown on the diagram.



   b) What do you notice about the denominators?

   c) What do you notice about the numerators of the *x*-coordinates? Compare them with the numerators of the *y*-coordinates. Why do these patterns make sense?

   d) Why are square roots involved?

   e) Explain this memory trick to a partner.

12. a) Explain, with reference to the unit circle, what the interval $-2\pi \leq \theta < 4\pi$ represents.

   b) Use your explanation to determine all values for θ in the interval $-2\pi \leq \theta < 4\pi$ such that $P(\theta) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

   c) How do your answers relate to the word "coterminal"?

13. If $P(\theta) = \left(-\frac{1}{3}, -\frac{2\sqrt{2}}{3}\right)$, determine the following.

   a) What does P(θ) represent? Explain using a diagram.

   b) In which quadrant does θ terminate?

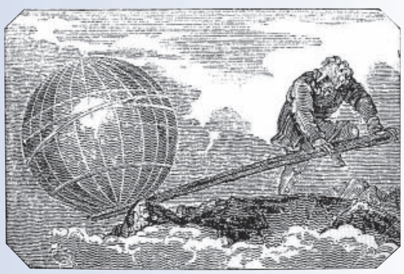
   c) Determine the coordinates of $P\left(\theta + \frac{\pi}{2}\right)$.

   d) Determine the coordinates of $P\left(\theta - \frac{\pi}{2}\right)$.

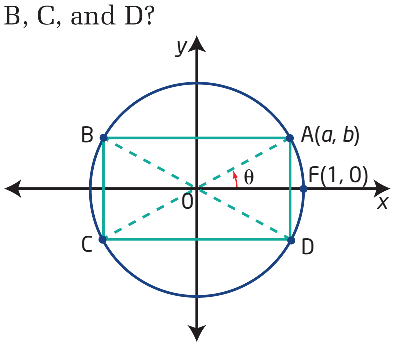
14. In ancient times, determining the perimeter and area of a circle were considered major mathematical challenges. One of Archimedes' greatest contributions to mathematics was his method for approximating π. Now, it is your turn to be a mathematician. Using a unit circle diagram, show the difference between π units and π square units.

Did You Know?

Archimedes was a Greek mathematician, physicist, inventor, and astronomer who lived from 287 BCE–212 BCE. He died in the Roman siege of Syracuse. He is considered one of the greatest mathematicians of all time. He correctly determined the value of π as being between $\frac{22}{7}$ and $\frac{223}{71}$ and proved the area of a circle to be πr^2, where r is the radius.



15. a) In the diagram, A has coordinates (a, b). ABCD is a rectangle with sides parallel to the axes. What are the coordinates of B, C, and D?



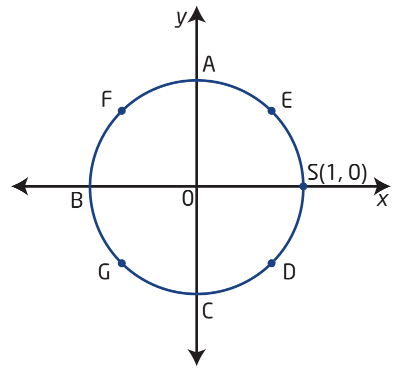
b) ∠FOA = θ, and A, B, C, and D lie on the unit circle. Through which point will the terminal arm pass for each angle? Assume all angles are in standard position.

i) $\theta + \pi$ **ii)** $\theta - \pi$

iii) $-\theta + \pi$ **iv)** $-\theta - \pi$

c) How are the answers in part b) different if θ is given as the measure of arc FA?

16. Use the unit circle diagram to answer the following questions. Points E, F, G, and D are midway between the axes.



a) What angle of rotation creates arc SG? What is the arc length of SG?

b) Which letter on the diagram corresponds to $P\left(\frac{13\pi}{2}\right)$? Explain your answer fully so someone not taking this course would understand. Use a diagram and a written explanation.

c) Between which two points would you find P(5)? Explain.

Extend

17. a) Determine the coordinates of all points where the line represented by $y = -3x$ intersects the unit circle. Give your answers as exact values in simplest form.

b) If one of the points is labelled $P(\theta + \pi)$, draw a diagram and show at least two values for θ. Explain what θ represents.

18. a) P(θ) lies at the intersection of the unit circle and the line joining A(5, 2) to the origin. Use your knowledge of similar triangles and the unit circle to determine the exact coordinates of P(θ).

b) Determine the radius of a larger circle with centre at the origin and passing through point A.

c) Write the equation for this larger circle.

19. In previous grades, you used sine and cosine as trigonometric ratios of sides of right triangles. Show how that use of trigonometry relates to the unit circle. Specifically, show that the coordinates of P(θ) can be represented by (cos θ, sin θ) for any θ in the unit circle.

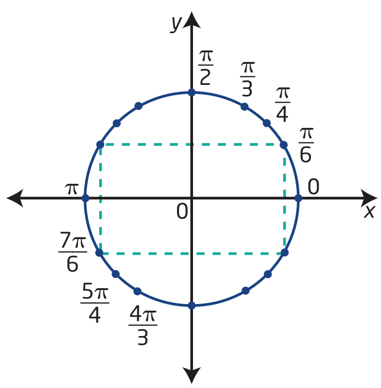
20. You can locate a point in a plane using Cartesian coordinates (x, y), where $|x|$ is the distance from the y-axis and $|y|$ is the distance from the x-axis. You can also locate a point in a plane using (r, θ), where r, $r \geq 0$, is the distance from the origin and θ is the angle of rotation from the positive x-axis. These are known as polar coordinates. Determine the polar coordinates for each point.

a) $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ **b)** $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{3}\right)$

c) $(2, 2)$ **d)** $(4, -3)$

Create Connections

C1 The diagram represents the unit circle with some positive arc lengths shown.



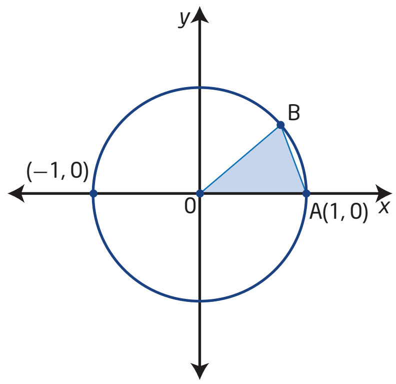
a) Draw a similar diagram in your notebook. Complete the labelling for positive measures.

b) Write the corresponding negative value beside each positive value. Complete this process over the interval $-2\pi \leq \theta < 0$.

c) Give the exact coordinates for the vertices of the dashed rectangle.

d) Identify several patterns from your unit circle diagrams. Patterns can relate to arc lengths, coordinates of points, and symmetry.

C2 Consider the isosceles △AOB drawn in the unit circle.



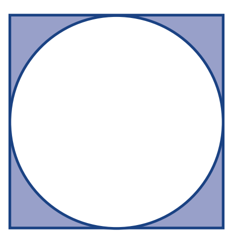
a) If the measure of one of the equal angles is twice the measure of the third angle, determine the exact measure of arc AB.

b) Draw a new △COA in which $P(C) = P\left(B + \frac{\pi}{2}\right)$. What is the exact measure of ∠CAO, in radians?

C3 **a)** Draw a diagram of a circle with centre at the origin and radius r units. What is the equation of this circle?

b) Show that the equation of any circle with centre (h, k) and radius r can be expressed as $(x - h)^2 + (y - k)^2 = r^2$. Hint: Use transformations to help with your explanation.

C4 The largest possible unit circle is cut from a square piece of paper as shown.



a) What percent of the paper is cut off? Give your answer to one decimal place.

b) What is the ratio of the circumference of the circle to the perimeter of the original piece of paper?

4.3

Trigonometric Ratios

Focus on…

- relating the trigonometric ratios to the coordinates of points on the unit circle
- determining exact and approximate values for trigonometric ratios
- identifying the measures of angles that generate specific trigonometric values
- solving problems using trigonometric ratios

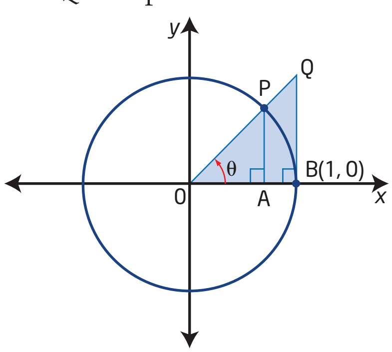
What do a software designer, a civil engineer, an airline pilot, and a long-distance swimmer's support team have in common? All of them use angles and trigonometric ratios to help solve problems. The software designer uses trigonometry to present a 3-D world on a 2-D screen. The engineer uses trigonometric ratios in designs of on-ramps and off-ramps at highway interchanges. A pilot uses an approach angle that is determined based on the tangent ratio. The support team for a long-distance swimmer uses trigonometry to compensate for the effect of wind and currents and to guide the swimmer's direction.

Investigate Trigonometric Ratios and the Unit Circle

Materials

- grid paper
- straight edge
- compass

1. Draw a unit circle as shown, with a positive angle θ in standard position. Work with a partner to describe the location of points P and Q. Be specific.



2. From your drawing, identify a single line segment whose length is equivalent to sin θ. Hint: Use the ratio definition of sin θ and the unit circle to help you.

3. Identify a line segment whose length is equivalent to cos θ and a line segment whose length is equivalent to tan θ in your diagram.

4. From your answers in steps 2 and 3, what could you use to represent the coordinates of point P?

Reflect and Respond

5. Present an argument or proof that the line segment you selected in step 3 for cos θ is correct.

6. What equation relates the coordinates of point P? Does this apply to any point P that lies at the intersection of the terminal arm for an angle θ and the unit circle? Why?

7. What are the maximum and minimum values for cos θ and sin θ? Express your answer in words and using an inequality. Confirm your answer using a calculator.

8. The value of tan θ changes from 0 to undefined for positive values of θ less than or equal to 90°. Explain how this change occurs with reference to angle θ in quadrant I of the unit circle. What happens on a calculator when tan θ is undefined?

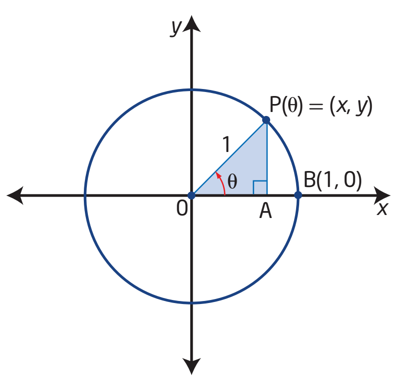
Link the Ideas

Coordinates in Terms of Primary Trigonometric Ratios

If P(θ) = (x, y) is the point on the terminal arm of angle θ that intersects the unit circle, notice that

- $\cos \theta = \frac{x}{1} = x$, which is the first coordinate of P(θ)
- $\sin \theta = \frac{y}{1} = y$, which is the second coordinate of P(θ)

How do these ratios connect to the right-triangle definition for cosine and sine?



You can describe the coordinates of any point P(θ) as (cos θ, sin θ). This is true for any point P(θ) at the intersection of the terminal arm of an angle θ and the unit circle.

Also, you know that $\tan \theta = \frac{y}{x}$.

Explain how this statement is consistent with the right-triangle definition of the tangent ratio.

Reciprocal Trigonometric Ratios

Three other trigonometric ratios are defined: they are the reciprocals of sine, cosine, and tangent. These are **cosecant**, **secant**, and **cotangent**.

By definition, $\csc\theta = \dfrac{1}{\sin\theta}$, $\sec\theta = \dfrac{1}{\cos\theta}$, and $\cot\theta = \dfrac{1}{\tan\theta}$.

cosecant ratio

- the reciprocal of the sine ratio
- abbreviated csc
- for $P(\theta) = (x, y)$ on the unit circle, $\csc\theta = \dfrac{1}{y}$
- if $\sin\theta = -\dfrac{\sqrt{3}}{2}$, then $\csc\theta = -\dfrac{2}{\sqrt{3}}$ or $-\dfrac{2\sqrt{3}}{3}$

secant ratio

- the reciprocal of the cosine ratio
- abbreviated sec
- for $P(\theta) = (x, y)$ on the unit circle, $\sec\theta = \dfrac{1}{x}$
- if $\cos\theta = \dfrac{1}{2}$, then $\sec\theta = \dfrac{2}{1}$ or 2

cotangent ratio

- the reciprocal of the tangent ratio
- abbreviated cot
- for $P(\theta) = (x, y)$ on the unit circle, $\cot\theta = \dfrac{x}{y}$
- if $\tan\theta = 0$, then $\cot\theta$ is undefined

Example 1

Determine the Trigonometric Ratios for Angles in the Unit Circle

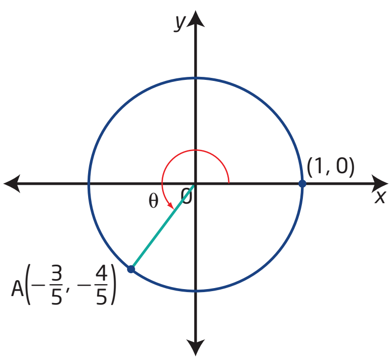
The point $A\left(-\dfrac{3}{5}, -\dfrac{4}{5}\right)$ lies at the intersection of the unit circle and the terminal arm of an angle θ in standard position.

a) Draw a diagram to model the situation.

b) Determine the values of the six trigonometric ratios for θ. Express answers in lowest terms.

Solution

a)



b) $\sin\theta = -\dfrac{4}{5}$ — The y-coordinate of $P(\theta)$ is defined as $\sin\theta$.

$\cos\theta = -\dfrac{3}{5}$ — Why is this true?

$$\tan\theta = \frac{y}{x}$$

$$= \frac{-\frac{4}{5}}{-\frac{3}{5}}$$

Explain the arithmetic used to simplify this double fraction.

$$= \frac{4}{3}$$

Why does it make sense for $\tan\theta$ to be positive?

$$\csc\theta = \frac{1}{\sin\theta}$$

$$= -\frac{5}{4}$$

Explain how this answer was determined.

$$\sec\theta = \frac{1}{\cos\theta}$$

Read as "sec θ equals the reciprocal of cos θ."

$$= -\frac{5}{3}$$

$$\cot\theta = \frac{1}{\tan\theta}$$

$$= \frac{3}{4}$$

Your Turn

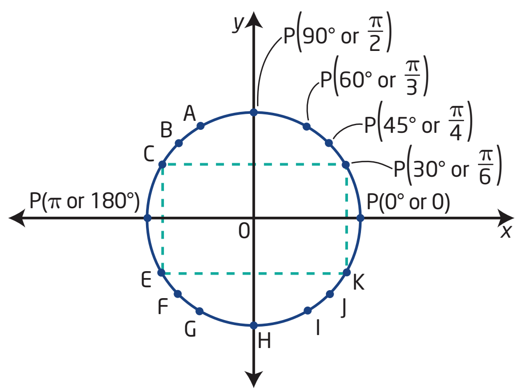
The point $B\left(-\frac{1}{3}, \frac{2\sqrt{2}}{3}\right)$ lies at the intersection of the unit circle and the terminal arm of an angle θ in standard position.

a) Draw a diagram to model the situation.

b) Determine the values of the six trigonometric ratios for θ. Express your answers in lowest terms.

Exact Values of Trigonometric Ratios

Exact values for the trigonometric ratios can be determined using special triangles (30°-60°-90° or 45°-45°-90°) and multiples of $\theta = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$, and $\frac{\pi}{2}$ or $\theta = 0°, 30°, 45°, 60°$, and $90°$ for points P(θ) on the unit circle.



How are P(30°), C, E, and K related?

What points have the same coordinates as $P\left(\frac{\pi}{3}\right)$ except for their signs?

For P(45°), what are the coordinates and in which quadrant is θ?

Which special triangle would you use and where would it be placed for θ = 135°?

Example 2

Exact Values for Trigonometric Ratios

Determine the exact value for each. Draw diagrams to illustrate your answers.

a) $\cos \frac{5\pi}{6}$

b) $\sin\left(-\frac{4\pi}{3}\right)$

c) sec 315°

d) cot 270°

Solution

a) The point $P\left(\frac{5\pi}{6}\right)$ lies in quadrant II.

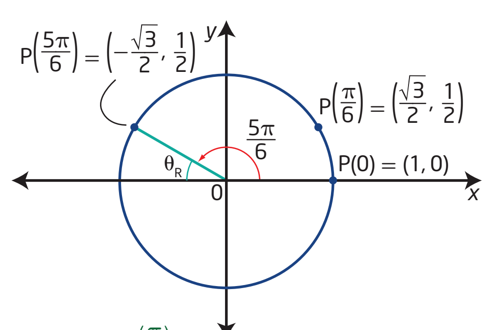
The reference angle for $\frac{5\pi}{6}$ is

$\theta_R = \pi - \frac{5\pi}{6} = \frac{\pi}{6}$.

Its x-coordinate is negative and its y-coordinate is positive.

$$P(\theta) = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$$

$$\cos \frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$$



$P\left(\frac{5\pi}{6}\right)$ has the same coordinates as $P\left(\frac{\pi}{6}\right)$, except the x-coordinates have different signs.

Recall that the reference angle, θ_R, is the acute angle formed between the terminal arm and the x-axis.

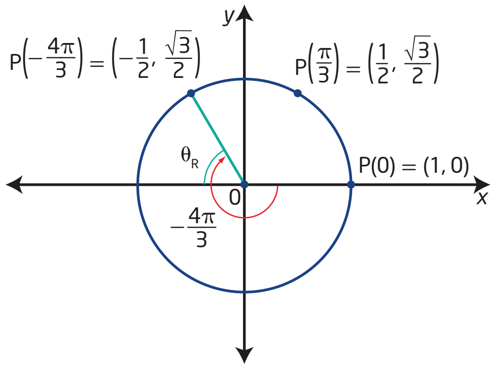
b) $-\frac{4\pi}{3}$ is a clockwise rotation from the positive x-axis.

$P\left(-\frac{4\pi}{3}\right)$ lies in quadrant II.

The reference angle for $-\frac{4\pi}{3}$ is $\theta_R = \pi - \frac{2\pi}{3} = \frac{\pi}{3}$.

$$P(\theta) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

$$\sin\left(-\frac{4\pi}{3}\right) = \frac{\sqrt{3}}{2}$$



What is a positive coterminal angle for $-\frac{4\pi}{3}$?

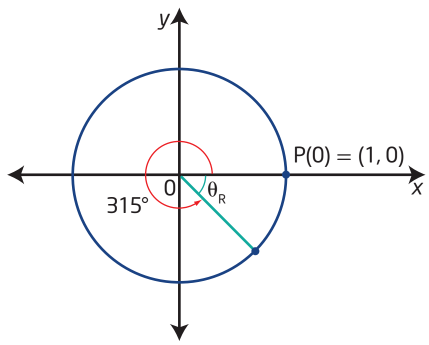
c) An angle of 315° is a counterclockwise rotation that terminates in quadrant IV.

The reference angle for 315° is $\theta_R = 360° - 315° = 45°$.

$$P(\theta) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

$$\sec 315° = \frac{1}{\cos 315°}$$
$$= \frac{\sqrt{2}}{1} \text{ or } \sqrt{2}$$

Explain how to get the coordinates for $P(\theta)$.



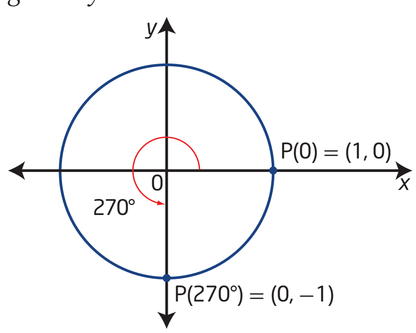
d) An angle of 270° terminates on the negative y-axis.

$P(270°) = (0, -1)$

Since $\tan\theta = \frac{y}{x}$, $\cot\theta = \frac{x}{y}$.

Therefore,

$$\cot 270° = \frac{0}{-1}$$
$$= 0$$



Your Turn

Draw diagrams to help you determine the exact value of each trigonometric ratio.

a) $\tan\frac{\pi}{2}$

b) $\csc\frac{7\pi}{6}$

c) $\sin(-300°)$

d) $\sec 60°$

Approximate Values of Trigonometric Ratios

You can determine approximate values for sine, cosine, and tangent using a scientific or graphing calculator. Most calculators can determine trigonometric values for angles measured in degrees or radians. You will need to set the mode to the correct angle measure. Check using

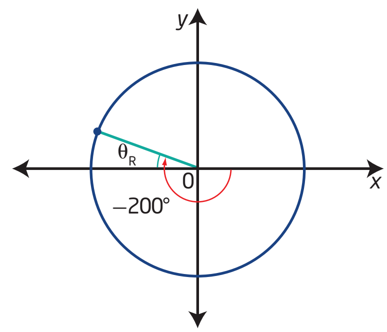
$\cos 60° = 0.5$ (degree mode)
$\cos 60 = -0.952\ 412\ 980\ldots$ (radian mode)

In which quadrant does an angle of 60 terminate?

Most calculators can compute trigonometric ratios for negative angles. However, you should use your knowledge of reference angles and the signs of trigonometric ratios for the quadrant to check that your calculator display is reasonable.

$\cos(-200°) = -0.939\ 692\ 620\ldots$

Is the negative value appropriate? What is the reference angle for −200°? What other trigonometric ratio could you compute as a check?



You can find the value of a trigonometric ratio for cosecant, secant, or cotangent using the correct reciprocal relationship.

$$\begin{aligned}\sec 3.3 &= \frac{1}{\cos 3.3}\\ &= -1.012\ 678\ 973\ldots\\ &\approx -1.0127\end{aligned}$$

Example 3

Approximate Values for Trigonometric Ratios

Determine the approximate value for each trigonometric ratio. Give your answers to four decimal places.

a) $\tan \frac{7\pi}{5}$ **b)** $\cos 260°$
c) $\sin 4.2$ **d)** $\csc(-70°)$

Solution

a) $\frac{7\pi}{5}$ is measured in radians.

In which quadrant does an angle of $\frac{7\pi}{5}$ terminate?

$$\begin{aligned}\tan \frac{7\pi}{5} &= 3.077\ 683\ 537\ldots\\ &\approx 3.0777\end{aligned}$$

Make sure your calculator is in radian mode.
Why is the answer positive?

b) $$\begin{aligned}\cos 260° &= -0.173\ 648\ 177\ldots\\ &\approx -0.1736\end{aligned}$$

In which quadrant does 260° terminate?

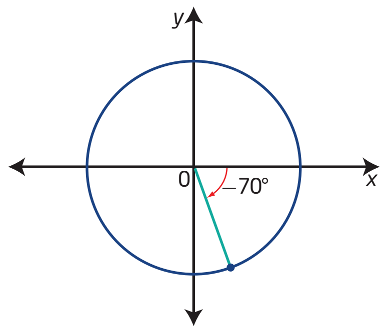
c) $\sin 4.2 = -0.871\ 575\ 772...$
≈ -0.8716

Which angle mode do you need here?
Why is the answer negative?

d) An angle of $-70°$ terminates in quadrant IV.
The y-coordinate for points in quadrant IV is negative.

$$\csc(-70°) = \frac{1}{\sin(-70°)}$$
$$= -1.064\ 177\ 772...$$
$$\approx -1.0642$$

What steps are needed to evaluate $\frac{1}{\sin(-70°)}$ on your calculator?



Your Turn

What is the approximate value for each trigonometric ratio? Round answers to four decimal places. Justify the sign of each answer.

a) $\sin 1.92$
b) $\tan(-500°)$
c) $\sec 85.4°$
d) $\cot 3$

Approximate Values of Angles

How can you find the measure of an angle when the value of the trigonometric ratio is given? To reverse the process (for example, to determine θ if you know sin θ), use the inverse trigonometric function keys on a calculator.

$\sin 30° = 0.5 \Longrightarrow \sin^{-1} 0.5 = 30°$

Note that $\sin^{-1}$ is an abbreviation for "the inverse of sine." Do not confuse this with $(\sin 30°)^{-1}$, which means $\frac{1}{\sin 30°}$, or the reciprocal of sin 30°.

The calculator keys $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$ return one answer only, when there are often two angles with the same trigonometric function value in any full rotation. In general, it is best to use the reference angle applied to the appropriate quadrants containing the terminal arm of the angle.

Example 4

Find Angles Given Their Trigonometric Ratios

Determine the measures of all angles that satisfy the following. Use diagrams in your explanation.

a) $\sin \theta = 0.879$ in the domain $0 \le \theta < 2\pi$. Give answers to the nearest tenth of a radian.

b) $\cos \theta = -0.366$ in the domain $0° \le \theta < 360°$. Give answers to the nearest tenth of a degree.

c) $\tan \theta = \sqrt{3}$ in the domain $-180° \le \theta < 180°$. Give exact answers.

d) $\sec \theta = \frac{2}{\sqrt{3}}$ in the domain $-2\pi \le \theta < 2\pi$. Give exact answers.

Solution

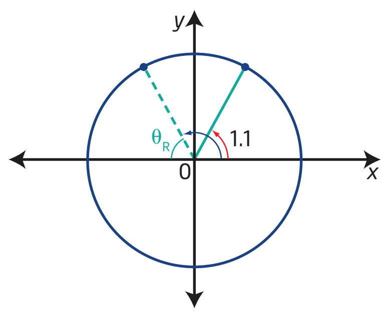
Did You Know?

By convention, if the domain is given in radian measure, express answers in radians. If the domain is expressed using degrees, give the answers in degrees.

a) $\sin \theta > 0$ in quadrants I and II.
The domain consists of one positive rotation.
Therefore, two answers need to be identified.

$\sin^{-1} 0.879 = 1.073\ 760\ 909\ldots$
≈ 1.1

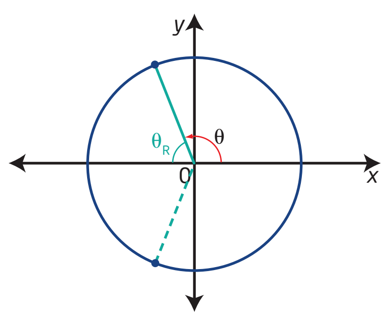
Use a calculator in radian mode.



In quadrant I, $\theta \approx 1.1$, to the nearest tenth. This is the reference angle.
In quadrant II, $\theta \approx \pi - 1.1$ or 2.0, to the nearest tenth.
The answers, to the nearest tenth of a radian, are 1.1 and 2.0.

b) $\cos \theta < 0$ in quadrants II and III.

Why will the answer be measured in degrees?



$\cos^{-1} (-0.366) \approx 111.5°$, to the nearest tenth.
This answer is in quadrant II.
The reference angle for other answers is 68.5°.
In quadrant III, $\theta \approx 180° + 68.5°$ or 248.5°.

Did you check that your calculator is in degree mode?

How do you determine this reference angle from 111.5°?

The answers, to the nearest tenth of a degree, are 111.5° and 248.5°.

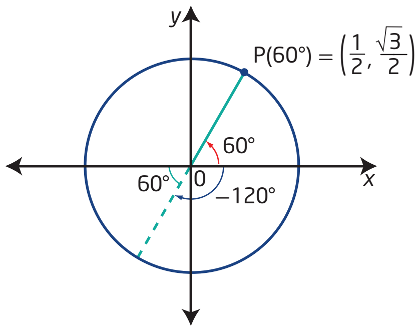
c) $\tan \theta > 0$ in quadrants I and III.

The domain includes both quadrants. In the positive direction an answer will be in quadrant I, and in the negative direction an answer will be in quadrant III.

To answer with exact values, work with the special coordinates on a unit circle.

$\tan 60° = \sqrt{3}$

How do you know that $\tan 60° = \sqrt{3}$? Could you use a calculator here?

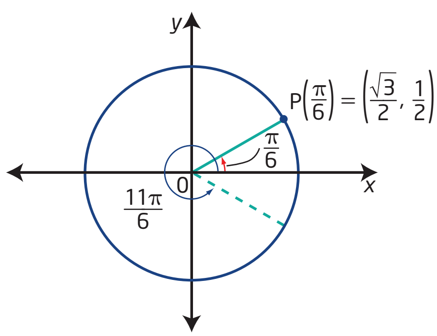


In quadrant I, from the domain $0° \leq \theta < 180°$, $\theta = 60°$. This is the reference angle. In quadrant III, from the domain $-180° \leq \theta < 0°$, $\theta = -180° + 60°$ or $-120°$.

The exact answers are 60° and −120°.

d) $\sec \theta > 0$ in quadrants I and IV since $\sec \theta = \frac{1}{\cos \theta}$ and $\cos \theta > 0$ in quadrants I and IV.

The domain includes four quadrants in both the positive and negative directions. Thus, there are two positive answers and two negative answers.



$\cos \theta = \frac{1}{\sec \theta} = \frac{\sqrt{3}}{2}$

$\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$\theta = \frac{\pi}{6}$ and $\theta = \frac{11\pi}{6}$ in the domain $0 \leq \theta < 2\pi$.

$\theta = -\frac{\pi}{6}$ and $\theta = -\frac{11\pi}{6}$ in the domain $-2\pi \leq \theta < 0$.

How do coterminal angles help?

The exact answers in radians are $\frac{\pi}{6}$, $\frac{11\pi}{6}$, $-\frac{\pi}{6}$, and $-\frac{11\pi}{6}$.

Your Turn

Determine the measures of all angles that satisfy each of the following. Use diagrams to show the possible answers.

a) $\cos\theta = 0.843$ in the domain $-360° < \theta < 180°$. Give approximate answers to the nearest tenth.

b) $\sin\theta = 0$ in the domain $0° \le \theta \le 180°$. Give exact answers.

c) $\cot\theta = -2.777$ in the domain $-\pi \le \theta \le \pi$. Give approximate answers to the nearest tenth.

d) $\csc\theta = -\dfrac{2}{\sqrt{2}}$ in the domain $-2\pi \le \theta \le \pi$. Give exact answers.

Example 5

Calculating Trigonometric Values for Points Not on the Unit Circle

The point A(−4, 3) lies on the terminal arm of an angle θ in standard position. What is the exact value of each trigonometric ratio for θ?

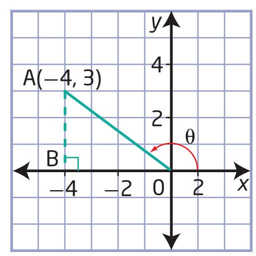
Solution

△ABO is a right triangle.

Identify trigonometric values for θ using the lengths of the sides of △ABO.

△ABO has sides of lengths 3, 4, and 5.

Confirm this using the Pythagorean theorem.



Recall that OA is a length and the segments OB and BA are considered as directed lengths.

$$\sin\theta = \frac{y}{r} = \frac{3}{5} \qquad \csc\theta = \frac{1}{\sin\theta} = \frac{5}{3}$$

$$\cos\theta = \frac{x}{r} = \frac{-4}{5} = -\frac{4}{5} \qquad \sec\theta = \frac{1}{\cos\theta} = -\frac{5}{4}$$

$$\tan\theta = \frac{y}{x} = \frac{3}{-4} = -\frac{3}{4} \qquad \cot\theta = \frac{1}{\tan\theta} = -\frac{4}{3}$$

Your Turn

The point D(−5, −12) lies on the terminal arm of an angle θ in standard position. What is the exact value of each trigonometric ratio for θ?

Key Ideas

- Points that are on the intersection of the terminal arm of an angle θ in standard position and the unit circle can be defined using trigonometric ratios.

  $P(\theta) = (\cos \theta, \sin \theta)$

- Each primary trigonometric ratio—sine, cosine, and tangent—has a reciprocal trigonometric ratio. The reciprocals are cosecant, secant, and cotangent, respectively.

  $\csc \theta = \dfrac{1}{\sin \theta}$ $\sec \theta = \dfrac{1}{\cos \theta}$ $\cot \theta = \dfrac{1}{\tan \theta}$

  If $\sin \theta = \frac{2}{3}$, then $\csc \theta = \frac{3}{2}$, and vice versa.

- You can determine the trigonometric ratios for any angle in standard position using the coordinates of the point where the terminal arm intersects the unit circle.
- Exact values of trigonometric rations for special angles such as 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, and $\frac{\pi}{2}$ and their multiples may be determined using the coordinates of points on the unit circle.
- You can determine approximate values for trigonometric ratios using a calculator in the appropriate mode: radians or degrees.
- You can use a scientific or graphing calculator to determine an angle measure given the value of a trigonometric ratio. Then, use your knowledge of reference angles, coterminal angles, and signs of ratios in each quadrant to determine other possible angle measures. Unless the domain is restricted, there are an infinite number of answers.
- Determine the trigonometric ratios for an angle θ in standard position from the coordinates of a point on the terminal arm of θ and right triangle definitions of the trigonometric ratios.

Check Your Understanding

Practise

1. What is the exact value for each trigonometric ratio?

a) $\sin 45°$ **b)** $\tan 30°$
c) $\cos \frac{3\pi}{4}$ **d)** $\cot \frac{7\pi}{6}$
e) $\csc 210°$ **f)** $\sec (-240°)$
g) $\tan \frac{3\pi}{2}$ **h)** $\sec \pi$
i) $\cot (-120°)$ **j)** $\cos 390°$
k) $\sin \frac{5\pi}{3}$ **l)** $\csc 495°$

2. Determine the approximate value for each trigonometric ratio. Give answers to two decimal places.

a) $\cos 47°$ **b)** $\cot 160°$
c) $\sec 15°$ **d)** $\csc 4.71$
e) $\sin 5$ **f)** $\tan 0.94$
g) $\sin \frac{5\pi}{7}$ **h)** $\tan 6.9$
i) $\cos 302°$ **j)** $\sin \left(-\frac{11\pi}{19}\right)$
k) $\cot 6$ **l)** $\sec (-270°)$

3. If θ is an angle in standard position with the following conditions, in which quadrants may θ terminate?

a) $\cos \theta > 0$

b) $\tan \theta < 0$

c) $\sin \theta < 0$

d) $\sin \theta > 0$ and $\cot \theta < 0$

e) $\cos \theta < 0$ and $\csc \theta > 0$

f) $\sec \theta > 0$ and $\tan \theta > 0$

4. Express the given quantity using the same trigonometric ratio and its reference angle. For example, $\cos 110° = -\cos 70°$. For angle measures in radians, give exact answers. For example, $\cos 3 = -\cos (\pi - 3)$.

a) $\sin 250°$ b) $\tan 290°$

c) $\sec 135°$ d) $\cos 4$

e) $\csc 3$ f) $\cot 4.95$

5. For each point, sketch two coterminal angles in standard position whose terminal arm contains the point. Give one positive and one negative angle, in radians, where neither angle exceeds one full rotation.

a) $(3, 5)$ b) $(-2, -1)$

c) $(-3, 2)$ d) $(5, -2)$

6. Indicate whether each trigonometric ratio is positive or negative. Do not use a calculator.

a) $\cos 300°$ b) $\sin 4$

c) $\cot 156°$ d) $\csc (-235°)$

e) $\tan \frac{13\pi}{6}$ f) $\sec \frac{17\pi}{3}$

7. Determine each value. Explain what the answer means.

a) $\sin^{-1} 0.2$ b) $\tan^{-1} 7$

c) $\sec 450°$ d) $\cot (-180°)$

8. The point $P(\theta) = \left(\frac{3}{5}, y\right)$ lies on the terminal arm of an angle θ in standard position and on the unit circle. $P(\theta)$ is in quadrant IV.

a) Determine y.

b) What is the value of $\tan \theta$?

c) What is the value of $\csc \theta$?

Apply

9. Determine the exact value of each expression.

a) $\cos 60° + \sin 30°$

b) $(\sec 45°)^2$

c) $\left(\cos \frac{5\pi}{3}\right)\left(\sec \frac{5\pi}{3}\right)$

d) $(\tan 60°)^2 - (\sec 60°)^2$

e) $\left(\cos \frac{7\pi}{4}\right)^2 + \left(\sin \frac{7\pi}{4}\right)^2$

f) $\left(\cot \frac{5\pi}{6}\right)^2$

10. Determine the exact measure of all angles that satisfy the following. Draw a diagram for each.

a) $\sin \theta = -\frac{1}{2}$ in the domain $0 \le \theta < 2\pi$

b) $\cot \theta = 1$ in the domain $-\pi \le \theta < 2\pi$

c) $\sec \theta = 2$ in the domain $-180° \le \theta < 90°$

d) $(\cos \theta)^2 = 1$ in the domain $-360° \le \theta < 360°$

11. Determine the approximate measure of all angles that satisfy the following. Give answers to two decimal places. Use diagrams to show the possible answers.

a) $\cos \theta = 0.42$ in the domain $-\pi \le \theta \le \pi$

b) $\tan \theta = -4.87$ in the domain $-\frac{\pi}{2} \le \theta \le \pi$

c) $\csc \theta = 4.87$ in the domain $-360° \le \theta < 180°$

d) $\cot \theta = 1.5$ in the domain $-180° \le \theta < 360°$

12. Determine the exact values of the other five trigonometric ratios under the given conditions.

a) $\sin \theta = \frac{3}{5}, \frac{\pi}{2} < \theta < \pi$

b) $\cos \theta = \frac{-2\sqrt{2}}{3}, -\pi \le \theta \le \frac{3\pi}{2}$

c) $\tan \theta = \frac{2}{3}, -360° < \theta < 180°$

d) $\sec \theta = \frac{4\sqrt{3}}{3}, -180° \le \theta \le 180°$

13. Using the point B(−2, −3), explain how to determine the exact value of cos θ given that B is a point on the terminal arm of an angle θ in standard position.

14. The measure of angle θ in standard position is 4900°.

a) Describe θ in terms of revolutions. Be specific.

b) In which quadrant does 4900° terminate?

c) What is the measure of the reference angle?

d) Give the value of each trigonometric ratio for 4900°.

15. a) Determine the positive value of $\sin(\cos^{-1} 0.6)$. Use your knowledge of the unit circle to explain why the answer is a rational number.

b) Without calculating, what is the positive value of $\cos(\sin^{-1} 0.6)$? Explain.

16. a) Jason got an answer of 1.051 176 209 when he used a calculator to determine the value of $\sec \frac{40\pi}{7}$. Is he correct? If not, where did he make his mistake?

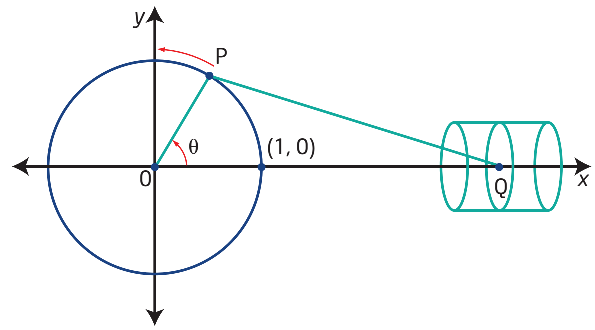
b) Describe the steps you would use to determine an approximate value for $\sec \frac{40\pi}{7}$ on your calculator.

17. a) Arrange the following values of sine in increasing order.
sin 1, sin 2, sin 3, sin 4

b) Show what the four values represent on a diagram of the unit circle. Use your diagram to justify the order from part a).

c) Predict the correct increasing order for cos 1, cos 2, cos 3, and cos 4. Check with a calculator. Was your prediction correct?

18. Examine the diagram. A piston rod, PQ, is connected to a wheel at P and to a piston at Q. As P moves around the wheel in a counterclockwise direction, Q slides back and forth.



a) What is the maximum distance that Q can move?

b) If the wheel starts with P at (1, 0) and rotates at 1 radian/s, draw a sketch to show where P will be located after 1 min.

c) What distance will Q have moved 1 s after start-up? Give your answer to the nearest hundredth of a unit.

19. Each point lies on the terminal arm of an angle θ in standard position. Determine θ in the specified domain. Round answers to the nearest hundredth of a unit.

a) A(−3, 4), $0 < \theta \le 4\pi$

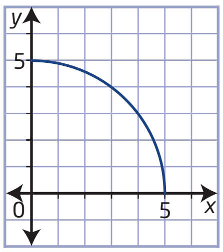
b) B(5, −1), $-360° \le \theta < 360°$

c) C(−2, −3), $-\frac{3\pi}{2} < \theta < \frac{7\pi}{2}$

Extend

20. Draw △ABC with ∠A = 15° and ∠C = 90°. Let BC = 1. D is a point on AC such that ∠DBC = 60°. Use your diagram to help you show that $\tan 15° = \frac{1}{\sqrt{3} + 2}$.

21. The diagram shows a quarter-circle of radius 5 units. Consider the point on the curve where $x = 2.5$. Show that this point is one-third the distance between (0, 5) and (5, 0) on the arc of the circle.



22. *Alice Through the Looking Glass* by Lewis Carroll introduced strange new worlds where time ran backwards. Your challenge is to imagine a unit circle in which a positive rotation is defined to be clockwise. Assume the coordinate system remains as we know it.

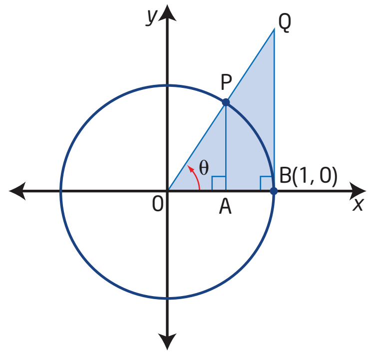
a) Draw a unit circle in which positive angles are measured clockwise from (0, 1). Label where $R\left(\frac{\pi}{6}\right)$, $R\left(\frac{5\pi}{6}\right)$, $R\left(\frac{7\pi}{6}\right)$, and $R\left(\frac{11\pi}{6}\right)$ are on your new unit circle.

b) What are the coordinates for the new $R\left(\frac{\pi}{6}\right)$ and $R\left(\frac{5\pi}{6}\right)$?

c) How do angles in this new system relate to conventional angles in standard position?

d) How does your new system of angle measure relate to bearings in navigation? Explain.

23. In the investigation at the beginning of this section, you identified line segments whose lengths are equivalent to cos θ, sin θ, and tan θ using the diagram shown.



a) Determine a line segment from the diagram whose length is equivalent to sec θ. Explain your reasoning

b) Make a copy of the diagram. Draw a horizontal line tangent to the circle that intersects the positive *y*-axis at C and OQ at D. Now identify segments whose lengths are equivalent to csc θ and cot θ. Explain your reasoning.

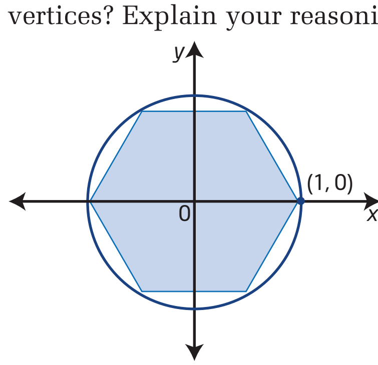
Create Connections

C1 a) Paula sees that sine ratios increase from 0 to 1 in quadrant 1. She concludes that the sine relation is increasing in quadrant I. Show whether Paula is correct using specific values for sine.

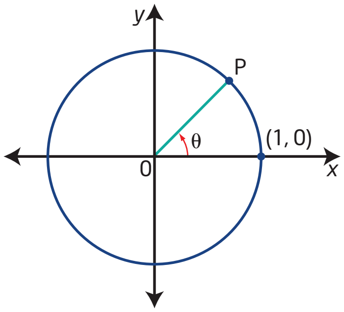
b) Is sine increasing in quadrant II? Explain why or why not.

c) Does the sine ratio increase in any other quadrant, and if so, which? Explain.

C2 A regular hexagon is inscribed in the unit circle as shown. If one vertex is at (1, 0), what are the exact coordinates of the other vertices? Explain your reasoning.



C3 Let P be the point of intersection of the unit circle and the terminal arm of an angle θ in standard position.



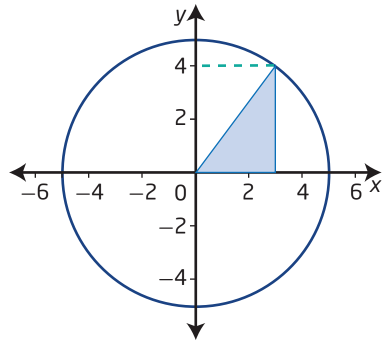
a) What is a formula for the slope of OP? Write your formula in terms of trigonometric ratios.

b) Does your formula apply in every quadrant? Explain.

c) Write an equation for any line OP. Use your trigonometric value for the slope.

d) Use transformations to show that your equation from part c) applies to any line where the slope is defined.

C4 Use the diagram to help find the value of each expression.



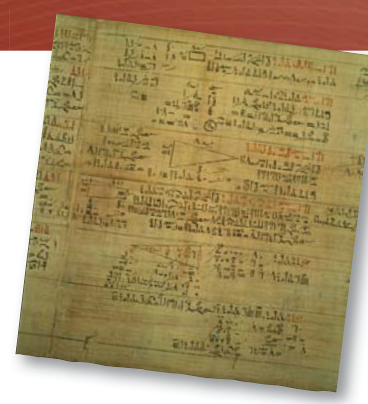
a) $\sin\left(\sin^{-1}\left(\frac{4}{5}\right)\right)$

b) $\cos\left(\tan^{-1}\left(\frac{4}{3}\right)\right)$

c) $\csc\left(\cos^{-1}\left(-\frac{3}{5}\right)\right)$, where the angle is in quadrant II

d) $\sin\left(\tan^{-1}\left(-\frac{4}{3}\right)\right)$, where the angle is in quadrant IV

Project Corner — History of Angle Measurement



Rhind Papyrus, ancient Egypt c1650 B.C.E.

- The use of the angular measurement unit "degree" is believed to have originated with the Babylonians. One theory is that their division of a circle into 360 parts is approximately the number of days in a year.
- Degree measures can also be subdivided into minutes (′) and seconds (″), where one degree is divided into 60 min, and one minute is divided into 60 s. For example, $30.1875° = 30°\ 11'\ 15''$.
- The earliest textual evidence of π dates from about 2000 B.C.E., with recorded approximations by the Babylonians $\left(\frac{25}{8}\right)$ and the Egyptians $\left(\frac{256}{81}\right)$. Roger Cotes (1682–1716) is credited with the concept of radian measure of angles, although he did not name the unit.
- The radian is widely accepted as the standard unit of angular measure in many fields of mathematics and in physics. The use of radians allows for the simplification of formulas and provides better approximations.
- What are some alternative units for measuring angles? What are some advantages and disadvantages of these units? What are some contexts in which these units are used?

4.4

Introduction to Trigonometric Equations

Focus on...

- algebraically solving first-degree and second-degree trigonometric equations in radians and in degrees
- verifying that a specific value is a solution to a trigonometric equation
- identifying exact and approximate solutions of a trigonometric equation in a restricted domain
- determining the general solution of a trigonometric equation

This old silver-gelatin photograph of traditional Kwakiutl spear fishing was taken in 1914 by Edward S. Curtis. The Kwakiutl First Nation's people have lived on the north-eastern shores of Vancouver Island for thousands of years. Today, the band council is based in Fort Rupert and owns 295 hectares of land in the area.

Many situations in nature involve cyclical patterns, such as average daily temperature at a specific location. Other applications of trigonometry relate to electricity or the way light passes from air into water. When you look at a fish in the water, it is not precisely where it appears to be, due to the refraction of light. The Kwakiutl peoples from Northwest British Columbia figured this out centuries ago. They became expert spear fishermen.

In this section, you will explore how to use algebraic techniques, similar to those used in solving linear and quadratic equations, to solve **trigonometric equations**. Your knowledge of coterminal angles, points on the unit circle, and inverse trigonometric functions will be important for understanding the solution of trigonometric equations.

trigonometric equation

- an equation involving trigonometric ratios

Investigate Trigonometric Equations

Did You Know?

In equations, mathematicians often use the notation $\cos^2 \theta$. This means the same as $(\cos \theta)^2$.

1. What are the exact measures of θ if $\cos \theta = -\frac{1}{2}$, $0 \leq \theta < 2\pi$? How is the equation related to $2 \cos \theta + 1 = 0$?

2. What is the answer for step 1 if the domain is given as $0° \leq \theta < 360°$?

3. What are the approximate measures for θ if $3 \cos \theta + 1 = 0$ and the domain is $0 \leq \theta < 2\pi$?

4. Set up a T-chart like the one below. In the left column, show the steps you would use to solve the quadratic equation $x^2 - x = 0$. In the right column, show similar steps that will lead to the solution of the trigonometric equation $\cos^2 \theta - \cos \theta = 0$, $0 \leq \theta < 2\pi$.

Quadratic Equation	Trigonometric Equation

Reflect and Respond

5. How is solving the equations in steps 1 to 3 similar to solving a linear equation? How is it different? Use examples.
6. When solving a trigonometric equation, how do you know whether to give your answers in degrees or radians?
7. Identify similarities and differences between solving a quadratic equation and solving a trigonometric equation that is quadratic.

Link the Ideas

In the investigation, you explored solving trigonometric equations. Did you realize that in Section 4.3 you were already solving simple trigonometric equations? The same processes will be used each time you solve a trigonometric equation, and these processes are the same as those used in solving linear and quadratic equations.

The notation $[0, \pi]$ represents the interval from 0 to π inclusive and is another way of writing $0 \leq \theta \leq \pi$.

- $\theta \in (0, \pi)$ means the same as $0 < \theta < \pi$.
- $\theta \in [0, \pi)$ means the same as $0 \leq \theta < \pi$.

How would you show $-\pi < \theta \leq 2\pi$ using interval notation?

Example 1

Solve Trigonometric Equations

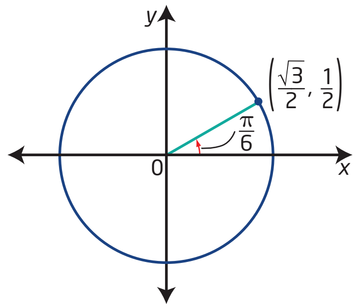
Solve each trigonometric equation in the specified domain.

a) $5 \sin \theta + 2 = 1 + 3 \sin \theta$, $0 \leq \theta < 2\pi$

b) $3 \csc x - 6 = 0$, $0° \leq x < 360°$

Solution

a)
$$
\begin{aligned}
5 \sin \theta + 2 &= 1 + 3 \sin \theta \\
5 \sin \theta + 2 - 3 \sin \theta &= 1 + 3 \sin \theta - 3 \sin \theta \\
2 \sin \theta + 2 &= 1 \\
2 \sin \theta + 2 - 2 &= 1 - 2 \\
2 \sin \theta &= -1 \\
\sin \theta &= -\frac{1}{2}
\end{aligned}
$$



The reference angle is $\theta_R = \frac{\pi}{6}$.

In which quadrants must θ terminate if $\sin \theta = -\frac{1}{2}$?

$\theta = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$ (quadrant III)

$\theta = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$ (quadrant IV)

The solutions are $\theta = \frac{7\pi}{6}$ and $\theta = \frac{11\pi}{6}$ in the domain $0 \le \theta < 2\pi$.

b) $3 \csc x - 6 = 0$

$3 \csc x = 6$

$\csc x = 2$ What operations were performed to arrive at this equation?

If $\csc x = 2$, then $\sin x = \frac{1}{2}$

$x = 30°$ and $150°$ Explain how to arrive at these answers.

The solutions are $x = 30°$ and $x = 150°$ in the domain $0° \le x < 360°$.

Your Turn

Solve each trigonometric equation in the specified domain.

a) $3 \cos \theta - 1 = \cos \theta + 1, -2\pi \le \theta \le 2\pi$

b) $4 \sec x + 8 = 0, 0° \le x < 360°$

Example 2

Factor to Solve a Trigonometric Equation

Solve for θ.

$\tan^2 \theta - 5 \tan \theta + 4 = 0, 0 \le \theta < 2\pi$

Give solutions as exact values where possible. Otherwise, give approximate angle measures, to the nearest thousandth of a radian.

Solution

$\tan^2 \theta - 5 \tan \theta + 4 = 0$ How is this similar to solving $x^2 - 5x + 4 = 0$?

$(\tan \theta - 1)(\tan \theta - 4) = 0$

$\tan \theta - 1 = 0$ or $\tan \theta - 4 = 0$

$\tan \theta = 1$ $\tan \theta = 4$ In which quadrants is $\tan \theta > 0$?

$\theta = \frac{\pi}{4}, \frac{5\pi}{4}$ $\tan^{-1} 4 = \theta$ What angle mode must your calculator be in to find $\tan^{-1} 4$?

$\theta = 1.3258...$

$\theta \approx 1.326$ is a measure in quadrant I. How do you know that 1.326 is in quadrant I?

In quadrant III,

$$\begin{aligned}\theta &= \pi + \theta_R\\ &= \pi + \tan^{-1} 4\\ &= \pi + 1.3258\ldots\\ &= 4.467\,410\,317\ldots\\ &\approx 4.467\end{aligned}$$

Why is $\tan^{-1} 4$ used as the reference angle here?

The solutions are $\theta = \frac{\pi}{4}$, $\theta = \frac{5\pi}{4}$ (exact), $\theta \approx 1.326$, and $\theta \approx 4.467$ (to the nearest thousandth).

Your Turn

Solve for θ.

$\cos^2 \theta - \cos \theta - 2 = 0$, $0° \leq \theta < 360°$

Give solutions as exact values where possible. Otherwise, give approximate measures to the nearest thousandth of a degree.

Example 3

General Solution of a Trigonometric Equation

a) Solve for x in the interval $0 \leq x < 2\pi$ if $\sin^2 x - 1 = 0$. Give answers as exact values.

b) Determine the general solution for $\sin^2 x - 1 = 0$ over the real numbers if x is measured in radians.

Solution

a) Method 1: Use Square Root Principles

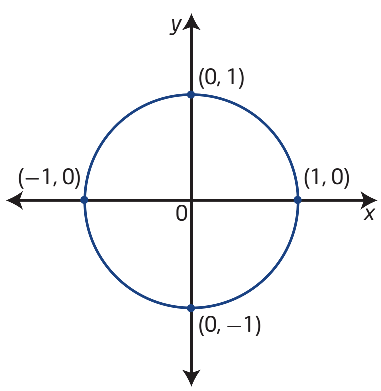
$$\begin{aligned}\sin^2 x - 1 &= 0\\ \sin^2 x &= 1\\ \sin x &= \pm 1\\ \sin x = 1 \text{ or } \sin x &= -1\end{aligned}$$

Why are there two values for $\sin x$?

If $\sin x = 1$, then $x = \frac{\pi}{2}$.

If $\sin x = -1$, then $x = \frac{3\pi}{2}$.

Where did $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ come from? Why is $\frac{5\pi}{2}$ not an acceptable answer?



Method 2: Use Factoring

$$\sin^2 x - 1 = 0$$
$$(\sin x - 1)(\sin x + 1) = 0$$
$$\sin x - 1 = 0 \quad \text{or} \quad \sin x + 1 = 0$$

Continue as in Method 1.

Check: $x = \frac{\pi}{2}$ | $x = \frac{3\pi}{2}$

Left Side	Right Side	Left Side	Right Side
$\sin^2 x - 1$	0	$\sin^2 x - 1$	0
$= \left(\sin \frac{\pi}{2}\right)^2 - 1$		$= \left(\sin \frac{3\pi}{2}\right)^2 - 1$	
$= 1^2 - 1$		$= (-1)^2 - 1$	
$= 0$		$= 0$	

Both answers are verified.

The solution is $x = \frac{\pi}{2}, \frac{3\pi}{2}$.

b) If the domain is real numbers, you can make an infinite number of rotations on the unit circle in both a positive and a negative direction.

Values corresponding to $x = \frac{\pi}{2}$ are $\ldots -\frac{7\pi}{2}, -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots$

What patterns do you see in these values for θ?

Values corresponding to $x = \frac{3\pi}{2}$ are $\ldots -\frac{5\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}, \ldots$

Do you see that the terminal arm is at the point (0, 1) or (0, −1) with any of the angles above?

Did You Know?

$2n$, where $n \in I$, represents all even integers.

$2n + 1$, where $n \in I$, is an expression for all odd integers.

An expression for the values corresponding to $x = \frac{\pi}{2}$ is $x = \frac{\pi}{2} + 2\pi n$, where $n \in I$.

An expression for the values corresponding to $x = \frac{3\pi}{2}$ is $x = \frac{3\pi}{2} + 2\pi n$, where $n \in I$.

The two expressions above can be combined to form the general solution $x = \frac{\pi}{2} + \pi n$, where $n \in I$.

The solution can also be described as "odd integral multiples of $\frac{\pi}{2}$." In symbols, this is written as $(2n + 1)\left(\frac{\pi}{2}\right)$, $n \in I$.

How can you show algebraically that $(2n + 1)\left(\frac{\pi}{2}\right)$, $n \in I$, and $\frac{\pi}{2} + \pi n$, $n \in I$, are equivalent?

Your Turn

a) If $\cos^2 x - 1 = 0$, solve for x in the domain $0° \le x < 360°$. Give solutions as exact values.

b) Determine the general solution for $\cos^2 x - 1 = 0$, where the domain is real numbers measured in degrees.

Key Ideas

- To solve a trigonometric equation algebraically, you can use the same techniques as used in solving linear and quadratic equations.
- When you arrive at $\sin \theta = a$ or $\cos \theta = a$ or $\tan \theta = a$, where $a \in \mathrm{R}$, then use the unit circle for exact values of θ and inverse trigonometric function keys on a calculator for approximate measures. Use reference angles to find solutions in other quadrants.
- To solve a trigonometric equation involving $\csc \theta$, $\sec \theta$, or $\cot \theta$, you may need to work with the related reciprocal value(s).
- To determine a general solution or if the domain is real numbers, find the solutions in one positive rotation (2π or 360°). Then, use the concept of coterminal angles to write an expression that identifies all possible measures.

Check Your Understanding

Practise

1. Without solving, determine the number of solutions for each trigonometric equation in the specified domain. Explain your reasoning.

a) $\sin \theta = \frac{\sqrt{3}}{2}$, $0 \leq \theta < 2\pi$

b) $\cos \theta = \frac{1}{\sqrt{2}}$, $-2\pi \leq \theta < 2\pi$

c) $\tan \theta = -1$, $-360° \leq \theta \leq 180°$

d) $\sec \theta = \frac{2\sqrt{3}}{3}$, $-180° \leq \theta < 180°$

2. The equation $\cos \theta = \frac{1}{2}$, $0 \leq \theta < 2\pi$, has solutions $\frac{\pi}{3}$ and $\frac{5\pi}{3}$. Suppose the domain is not restricted.

a) What is the general solution corresponding to $\theta = \frac{\pi}{3}$?

b) What is the general solution corresponding to $\theta = \frac{5\pi}{3}$?

3. Determine the exact roots for each trigonometric equation or statement in the specified domain.

a) $2 \cos \theta - \sqrt{3} = 0$, $0 \leq \theta < 2\pi$

b) $\csc \theta$ is undefined, $0° \leq \theta < 360°$

c) $5 - \tan^2 \theta = 4$, $-180° \leq \theta \leq 360°$

d) $\sec \theta + \sqrt{2} = 0$, $-\pi \leq \theta \leq \frac{3\pi}{2}$

4. Solve each equation for $0 \leq \theta < 2\pi$. Give solutions to the nearest hundredth of a radian.

a) $\tan \theta = 4.36$

b) $\cos \theta = -0.19$

c) $\sin \theta = 0.91$

d) $\cot \theta = 12.3$

e) $\sec \theta = 2.77$

f) $\csc \theta = -1.57$

5. Solve each equation in the specified domain.

a) $3 \cos \theta - 1 = 4 \cos \theta$, $0 \leq \theta < 2\pi$

b) $\sqrt{3} \tan \theta + 1 = 0$, $-\pi \leq \theta \leq 2\pi$

c) $\sqrt{2} \sin x - 1 = 0$, $-360° < x \leq 360°$

d) $3 \sin x - 5 = 5 \sin x - 4$, $-360° \leq x < 180°$

e) $3 \cot x + 1 = 2 + 4 \cot x$, $-180° < x < 360°$

f) $\sqrt{3} \sec \theta + 2 = 0$, $-\pi \leq \theta \leq 3\pi$

6. Copy and complete the table to express each domain or interval using the other notation.

	Domain	Interval Notation
a)	$-2\pi \le \theta \le 2\pi$	
b)	$-\frac{\pi}{3} \le \theta \le \frac{7\pi}{3}$	
c)	$0° \le \theta \le 270°$	
d)		$\theta \in [0, \pi)$
e)		$\theta \in (0°, 450°)$
f)		$\theta \in (-2\pi, 4\pi]$

7. Solve for θ in the specified domain. Give solutions as exact values where possible. Otherwise, give approximate measures to the nearest thousandth.

a) $2\cos^2\theta - 3\cos\theta + 1 = 0$, $0 \le \theta < 2\pi$

b) $\tan^2\theta - \tan\theta - 2 = 0$, $0° \le \theta < 360°$

c) $\sin^2\theta - \sin\theta = 0$, $\theta \in [0, 2\pi)$

d) $\sec^2\theta - 2\sec\theta - 3 = 0$, $\theta \in [-180°, 180°)$

8. Todd believes that 180° and 270° are solutions to the equation $5\cos^2\theta = -4\cos\theta$. Show how you would check to determine whether Todd's solutions are correct.

Apply

9. Aslan and Shelley are finding the solution for $2\sin^2\theta = \sin\theta$, $0 < \theta \le \pi$. Here is their work.

$2\sin^2\theta = \sin\theta$

$\frac{2\sin^2\theta}{\sin\theta} = \frac{\sin\theta}{\sin\theta}$ Step 1

$2\sin\theta = 1$ Step 2

$\sin\theta = \frac{1}{2}$ Step 3

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$ Step 4

a) Identify the error that Aslan and Shelley made and explain why their solution is incorrect.

b) Show a correct method to determine the solution for $2\sin^2\theta = \sin\theta$, $0 < \theta \le \pi$.

10. Explain why the equation $\sin\theta = 0$ has no solution in the interval $(\pi, 2\pi)$.

11. What is the solution for $\sin\theta = 2$? Show how you know. Does the interval matter?

12. Jaycee says that the trigonometric equation $\cos\theta = \frac{1}{2}$ has an infinite number of solutions. Do you agree? Explain.

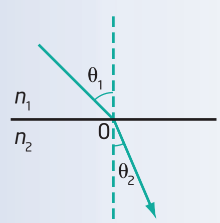
13. **a)** Helene is asked to solve the equation $3\sin^2\theta - 2\sin\theta = 0$, $0 \le \theta \le \pi$. She finds that $\theta = \pi$. Show how she could check whether this is a correct root for the equation.

b) Find all the roots of the equation $3\sin^2\theta - 2\sin\theta = 0$, $\theta \in [0, \pi]$.

14. Refer to the Did You Know? below. Use Snell's law of refraction to determine the angle of refraction of a ray of light passing from air into water if the angle of incidence is 35°. The refractive index is 1.000 29 for air and 1.33 for water.

Did You Know?

Willebrord Snell, a Dutch physicist, discovered that light is bent (refracted) as it passes from one medium into another. Snell's law is shown in the diagram.



$n_1 \sin\theta_1 = n_2 \sin\theta_2$,
where θ_1 is the angle of incidence,
θ_2 is the angle of refraction, and
n_1 and n_2 are the refractive indices of the mediums.

15. The average number of air conditioners sold in western Canada varies seasonally and depends on the month of the year. The formula $y = 5.9 + 2.4 \sin\left(\frac{\pi}{6}(t - 3)\right)$ gives the expected sales, y, in thousands, according to the month, t, where $t = 1$ represents January, $t = 2$ is February, and so on.

a) In what month are sales of 8300 air conditioners expected?

b) In what month are sales expected to be least?

c) Does this formula seem reasonable? Explain.

16. Nora is required to solve the following trigonometric equation.

$9 \sin^2 \theta + 12 \sin \theta + 4 = 0, \theta \in [0°, 360°)$

Nora did the work shown below. Examine her work carefully. Identify any errors. Rewrite the solution, making any changes necessary for it to be correct.

$$9 \sin^2 \theta + 12 \sin \theta + 4 = 0$$
$$(3 \sin \theta + 2)^2 = 0$$
$$3 \sin \theta + 2 = 0$$

Therefore, $\sin \theta = -\frac{2}{3}$

Use a calculator.

$$\sin^{-1}\left(-\frac{2}{3}\right) = -41.810\ 314\ 9$$

So, the reference angle is 41.8°, to the nearest tenth of a degree.

Sine is negative in quadrants II and III.

The solution in quadrant II is 180° − 41.8° = 138.2°.

The solution in quadrant III is 180° + 41.8° = 221.8°.

Therefore, θ = 138.2° and θ = 221.8°, to the nearest tenth of a degree.

17. Identify two different cases when a trigonometric equation would have no solution. Give an example to support each case.

18. Find the value of sec θ if $\cot \theta = \frac{3}{4}$, $180° \le \theta \le 270°$.

Extend

19. A beach ball is riding the waves near Tofino, British Columbia. The ball goes up and down with the waves according to the formula $h = 1.4 \sin\left(\frac{\pi t}{3}\right)$, where h is the height, in metres, above sea level, and t is the time, in seconds.

a) In the first 10 s, when is the ball at sea level?

b) When does the ball reach its greatest height above sea level? Give the first time this occurs and then write an expression for every time the maximum occurs.

c) According to the formula, what is the most the ball goes below sea level?

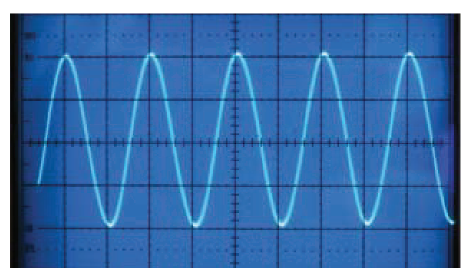
20. The current, I, in amperes, for an electric circuit is given by the formula $I = 4.3 \sin 120\pi t$, where t is time, in seconds.

a) The alternating current used in western Canada cycles 60 times per second. Demonstrate this using the given formula.

b) At what times is the current at its maximum value? How does your understanding of coterminal angles help in your solution?

c) At what times is the current at its minimum value?

d) What is the maximum current?



Oscilloscopes can measure wave functions of varying voltages.

21. Solve the trigonometric equation $\cos\left(x - \frac{\pi}{2}\right) = \frac{\sqrt{3}}{2}$, $-\pi < x < \pi$.

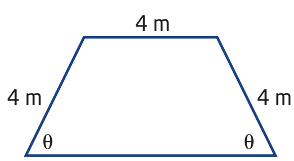
22. Consider the trigonometric equation $\sin^2 \theta + \sin \theta - 1 = 0$.

a) Can you solve the equation by factoring?

b) Use the quadratic formula to solve for $\sin \theta$.

c) Determine all solutions for θ in the interval $0 < \theta \leq 2\pi$. Give answers to the nearest hundredth of a radian, if necessary.

23. Jaime plans to build a new deck behind her house. It is to be an isosceles trapezoid shape, as shown. She would like each outer edge of the deck to measure 4 m.



a) Show that the area, A, of the deck is given by $A = 16 \sin \theta(1 + \cos \theta)$.

b) Determine the exact value of θ in radians if the area of the deck is $12\sqrt{3}$ m^2.

c) The angle in part b) gives the maximum area for the deck. How can you prove this? Compare your method with that of another student.

Create Connections

C1 Compare and contrast solving linear and quadratic equations with solving linear and quadratic trigonometric equations.

C2 A computer determines that a point on the unit circle has coordinates A(0.384 615 384 6, 0.923 076 923 1).

a) How can you check whether a point is on the unit circle? Use your method to see if A is on the unit circle.

b) If A is the point where the terminal arm of an angle θ intersects the unit circle, determine the values of $\cos \theta$, $\tan \theta$, and $\csc \theta$. Give your answers to three decimal places.

c) Determine the measure of angle θ, to the nearest tenth of a degree. Does this approximate measure for θ seem reasonable for point A? Explain using a diagram.

C3 Use your knowledge of non-permissible values for rational expressions to answer the following.

a) What is meant by the expression "non-permissible values"? Give an example.

b) Use the fact that any point on the unit circle has coordinates $P(\theta) = (\cos \theta, \sin \theta)$ to identify a trigonometric relation that could have non-permissible values.

c) For the trigonometric relation that you identified in part b), list all the values of θ in the interval $0 \leq \theta < 4\pi$ that are non-permissible.

d) Create a general statement for all the non-permissible values of θ for your trigonometric relation over the real numbers.

C4 **a)** Determine all solutions for the equation $2 \sin^2 \theta = 1 - \sin \theta$ in the domain $0° \leq \theta < 360°$.

b) Are your solutions exact or approximate? Why?

c) Show how you can check one of your solutions to verify its correctness.

Chapter 4 Review

4.1 Angles and Angle Measure, pages 166–179

1. If each angle is in standard position, in which quadrant does it terminate?

a) $100°$

b) $500°$

c) 10

d) $\frac{29\pi}{6}$

2. Draw each angle in standard position. Convert each degree measure to radian measure and each radian measure to degree measure. Give answers as exact values.

a) $\frac{5\pi}{2}$

b) $240°$

c) $-405°$

d) -3.5

3. Convert each degree measure to radian measure and each radian measure to degree measure. Give answers as approximate values to the nearest hundredth, where necessary.

a) $20°$

b) $-185°$

c) -1.75

d) $\frac{5\pi}{12}$

4. Determine the measure of an angle coterminal with each angle in the domain $0° \leq \theta < 360°$ or $0 \leq \theta < 2\pi$. Draw a diagram showing the quadrant in which each angle terminates.

a) 6.75

b) $400°$

c) -3

d) $-105°$

5. Write an expression for all of the angles coterminal with each angle. Indicate what your variable represents.

a) $250°$

b) $\frac{5\pi}{2}$

c) $-300°$

d) 6

6. A jet engine motor cycle is tested at 80 000 rpm. What is this angular velocity in

a) radians per minute?

b) degrees per second?

4.2 The Unit Circle, pages 180–190

7. $P(\theta) = (x, y)$ is the point where the terminal arm of an angle θ intersects the unit circle. What are the coordinates for each point?

a) $P\left(\frac{5\pi}{6}\right)$

b) $P(-150°)$

c) $P\left(-\frac{11\pi}{2}\right)$

d) $P(45°)$

e) $P(120°)$

f) $P\left(\frac{11\pi}{3}\right)$

8. a) If the coordinates for $P\left(\frac{\pi}{3}\right)$ are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, explain how you can determine the coordinates for $P\left(\frac{2\pi}{3}\right)$, $P\left(\frac{4\pi}{3}\right)$, and $P\left(\frac{5\pi}{3}\right)$.

   b) If the coordinates for $P(\theta)$ are $\left(-\frac{2\sqrt{2}}{3}, \frac{1}{3}\right)$, what are the coordinates for $P\left(\theta + \frac{\pi}{2}\right)$?

   c) In which quadrant does $P\left(\frac{5\pi}{6} + \pi\right)$ lie? Explain how you know. If $P\left(\frac{5\pi}{6} + \pi\right)$ represents $P(\theta)$, what is the measure of θ and what are the coordinates of $P(\theta)$?

9. Identify all measures for θ in the interval $-2\pi \leq \theta < 2\pi$ such that $P(\theta)$ is the given point.

   a) $(0, 1)$

   b) $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

   c) $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

   d) $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

10. Identify all measures for θ in the domain $-180° < \theta \leq 360°$ such that $P(\theta)$ is the given point.

   a) $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

   b) $(-1, 0)$

   c) $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

   d) $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

11. If $P(\theta) = \left(\frac{\sqrt{5}}{3}, -\frac{2}{3}\right)$, answer the following questions.

   a) What is the measure of θ? Explain using a diagram.

   b) In which quadrant does θ terminate?

   c) What are the coordinates of $P(\theta + \pi)$?

   d) What are the coordinates of $P\left(\theta + \frac{\pi}{2}\right)$?

   e) What are the coordinates of $P\left(\theta - \frac{\pi}{2}\right)$?

4.3 Trigonometric Ratios, pages 191–205

12. If $\cos \theta = \frac{1}{3}$, $0° \leq \theta \leq 270°$, what is the value of each of the other trigonometric ratios of θ? When radicals occur, leave your answer in exact form.

13. Without using a calculator, determine the exact value of each trigonometric ratio.

   a) $\sin \left(-\frac{3\pi}{2}\right)$

   b) $\cos \frac{3\pi}{4}$

   c) $\cot \frac{7\pi}{6}$

   d) $\sec (-210°)$

   e) $\tan 720°$

   f) $\csc 300°$

14. Determine the approximate measure of all angles that satisfy the following. Give answers to the nearest hundredth of a unit. Draw a sketch to show the quadrant(s) involved.

   a) $\sin \theta = 0.54$, $-2\pi < \theta \leq 2\pi$

   b) $\tan \theta = 9.3$, $-180° \leq \theta < 360°$

   c) $\cos \theta = -0.77$, $-\pi \leq \theta < \pi$

   d) $\csc \theta = 9.5$, $-270° < \theta \leq 90°$

15. Determine each trigonometric ratio, to three decimal places.

a) $\sin 285°$

b) $\cot 130°$

c) $\cos 4.5$

d) $\sec 7.38$

16. The terminal arm of an angle θ in standard position passes through the point $A(-3, 4)$.

a) Draw the angle and use a protractor to determine its measure, to the nearest degree.

b) Show how to determine the exact value of $\cos \theta$.

c) What is the exact value of $\csc \theta + \tan \theta$?

d) From the value of $\cos \theta$, determine the measure of θ in degrees and in radians, to the nearest tenth.

4.4 Introduction to Trigonometric Equations, pages 206–214

17. Factor each trigonometric expression.

a) $\cos^2 \theta + \cos \theta$

b) $\sin^2 \theta - 3 \sin \theta - 4$

c) $\cot^2 \theta - 9$

d) $2 \tan^2 \theta - 9 \tan \theta + 10$

18. Explain why it is impossible to find each of the following values.

a) $\sin^{-1} 2$

b) $\tan 90°$

19. Without solving, determine the number of solutions for each trigonometric equation or statement in the specified domain.

a) $4 \cos \theta - 3 = 0, 0° < \theta \leq 360°$

b) $\sin \theta + 0.9 = 0, -\pi \leq \theta \leq \pi$

c) $0.5 \tan \theta - 1.5 = 0, -180° \leq \theta \leq 0°$

d) $\csc \theta$ is undefined, $\theta \in [-2\pi, 4\pi)$

20. Determine the exact roots for each trigonometric equation.

a) $\csc \theta = \sqrt{2}, \theta \in [0°, 360°]$

b) $2 \cos \theta + 1 = 0, 0 \leq \theta < 2\pi$

c) $3 \tan \theta - \sqrt{3} = 0, -180° \leq \theta < 360°$

d) $\cot \theta + 1 = 0, -\pi \leq \theta < \pi$

21. Solve for θ. Give solutions as exact values where possible. Otherwise, give approximate measures, to the nearest thousandth.

a) $\sin^2 \theta + \sin \theta - 2 = 0, 0 \leq \theta < 2\pi$

b) $\tan^2 \theta + 3 \tan \theta = 0, 0° < \theta \leq 360°$

c) $6 \cos^2 \theta + \cos \theta = 1, \theta \in (0°, 360°)$

d) $\sec^2 \theta - 4 = 0, \theta \in [-\pi, \pi]$

22. Determine a domain for which the equation $\sin \theta = \frac{\sqrt{3}}{2}$ would have the following solution.

a) $\theta = \frac{\pi}{3}, \frac{2\pi}{3}$

b) $\theta = -\frac{5\pi}{3}, -\frac{4\pi}{3}, \frac{\pi}{3}$

c) $\theta = -660°, -600°, -300°, -240°$

d) $\theta = -240°, 60°, 120°, 420°$

23. Determine each general solution using the angle measure specified.

a) $\sin x = -\frac{1}{2}$, in radians

b) $\sin x = \sin^2 x$, in degrees

c) $\sec x + 2 = 0$, in degrees

d) $(\tan x - 1)(\tan x - \sqrt{3}) = 0$, in radians

Chapter 4 Practice Test

Multiple Choice

For #1 to #5, choose the best answer.

1. If $\cos \theta = \frac{\sqrt{3}}{2}$, which could be the measure of θ?

A $\frac{2\pi}{3}$ **B** $\frac{5\pi}{6}$ **C** $\frac{5\pi}{3}$ **D** $\frac{11\pi}{6}$

2. Which exact measures of θ satisfy $\sin \theta = -\frac{\sqrt{3}}{2}$, $0° \le \theta < 360°$?

A $60°, 120°$

B $-60°, -120°$

C $240°, 300°$

D $-240°, -300°$

3. If $\cot \theta = 1.4$, what is one approximate measure in radians for θ?

A 0.620

B 0.951

C 1.052

D 0.018

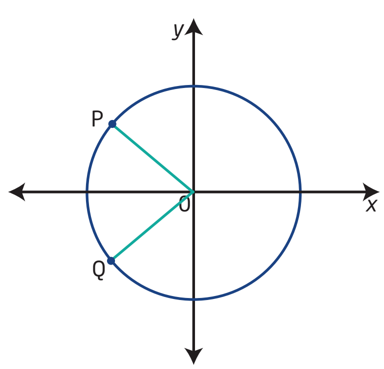
4. The coordinates of point P on the unit circle are $\left(-\frac{3}{4}, \frac{\sqrt{7}}{4}\right)$. What are the coordinates of Q if Q is a 90° counterclockwise rotation from P?

A $\left(\frac{\sqrt{7}}{4}, -\frac{3}{4}\right)$

B $\left(-\frac{\sqrt{7}}{4}, -\frac{3}{4}\right)$

C $\left(\frac{3}{4}, \frac{\sqrt{7}}{4}\right)$

D $\left(-\frac{3}{4}, -\frac{\sqrt{7}}{4}\right)$



5. Determine the number of solutions for the trigonometric equation $\sin \theta (\sin \theta + 1) = 0$, $-180° < \theta < 360°$.

A 3

B 4

C 5

D 6

Short Answer

6. A vehicle has tires that are 75 cm in diameter. A point is marked on the edge of the tire.

a) Determine the measure of the angle through which the point turns every second if the vehicle is travelling at 110 km/h. Give your answer in degrees and in radians, to the nearest tenth.

b) What is the answer in radians if the diameter of the tire is 66 cm? Do you think that tire diameter affects tire life? Explain.

7. a) What is the equation for any circle with centre at the origin and radius 1 unit?

b) Determine the value(s) for the missing coordinate for all points on the unit circle satisfying the given conditions. Draw diagrams.

i) $\left(\frac{2\sqrt{3}}{5}, y\right)$

ii) $\left(x, \frac{\sqrt{7}}{4}\right)$, $x < 0$

c) Explain how to use the equation for the unit circle to find the value of $\cos \theta$ if you know the y-coordinate of the point where the terminal arm of an angle θ in standard position intersects the unit circle.

8. Suppose that the cosine of an angle is negative and that you found one solution in quadrant III.

   a) Explain how to find the other solution between 0 and 2π.

   b) Describe how to write the general solution.

9. Solve the equation $2 \cos \theta + \sqrt{2} = 0$, where $\theta \in R$.

10. Explain the difference between an angle measuring 3° and one measuring 3 radians.

11. An angle in standard position measures −500°.

    a) In which quadrant does −500° terminate?

    b) What is the measure of the reference angle?

    c) What is the approximate value, to one decimal place, of each trigonometric ratio for −500°?

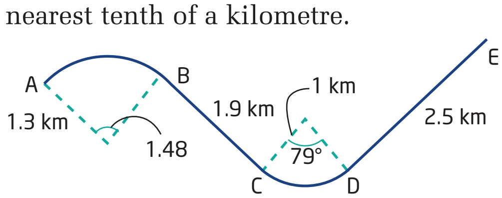
12. Identify one positive and one negative angle measure that is coterminal with each angle. Then, write a general expression for all the coterminal angles in each case.

    a) $\frac{13\pi}{4}$

    b) −575°

Extended Response

13. The diagram shows a stretch of road from A to E. The curves are arcs of circles. Determine the length of the road from A to E. Give your answer to the nearest tenth of a kilometre.



14. Draw any $\triangle ABC$ with A at the origin, side AB along the positive x-axis, and C in quadrant I. Show that the area of your triangle can be expressed as $\frac{1}{2}bc \sin A$ or $\frac{1}{2}ac \sin B$.

15. Solve for θ. Give solutions as exact values where possible. Otherwise, give approximate measures to the nearest hundredth.

    a) $3 \tan^2 \theta - \tan \theta - 4 = 0, -\pi < \theta < 2\pi$

    b) $\sin^2 \theta + \sin \theta - 1 = 0, 0 \leq \theta < 2\pi$

    c) $\tan^2 \theta = 4 \tan \theta, \theta \in [0, 2\pi]$

16. Jack chooses a horse to ride on the West Edmonton Mall carousel. The horse is located 8 m from the centre of the carousel. If the carousel turns through an angle of 210° before stopping to let a crying child get off, how far did Jack travel? Give your answer as both an exact value and an approximate measure to the nearest hundredth of a metre.