CHAPTER 2

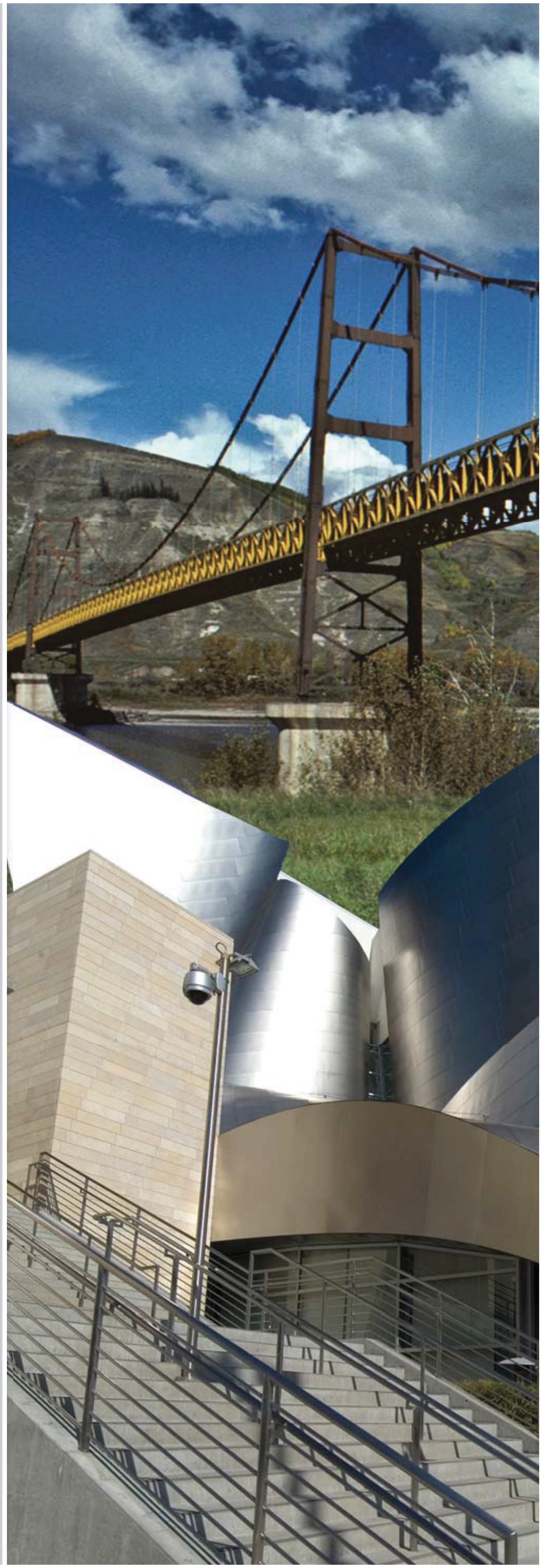
Trigonometry

Trigonometry has many applications. Bridge builders require an understanding of forces acting at different angles. Many bridges are supported by triangles. Trigonometry is used to design bridge side lengths and angles for maximum strength and safety.

Global positioning systems (GPSs) are used in many aspects of our lives, from cellphones and cars to mining and excavation. A GPS receiver uses satellites to triangulate a position, locating that position in terms of its latitude and longitude. Land surveying, energy conservation, and solar panel placement all require knowledge of angles and an understanding of trigonometry.

Using either the applications mentioned here or the photographs, describe three situations in which trigonometry could be used.

You may think of trigonometry as the study of acute angles and right triangles. In this chapter, you will extend your study of trigonometry to angles greater than 90° and to non-right triangles.



Did You Know?

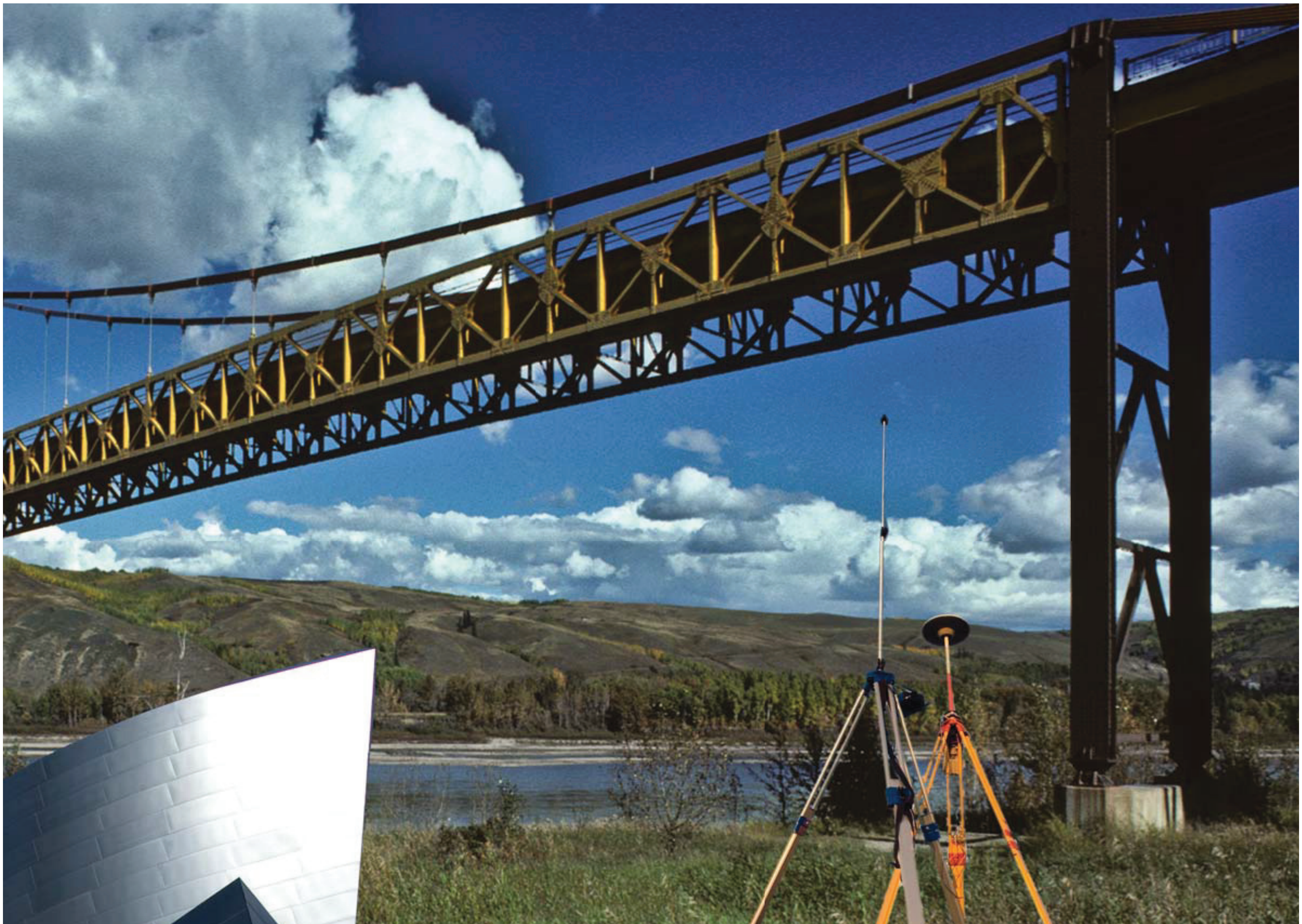
Euclid defined an angle in his textbook *The Elements* as follows:

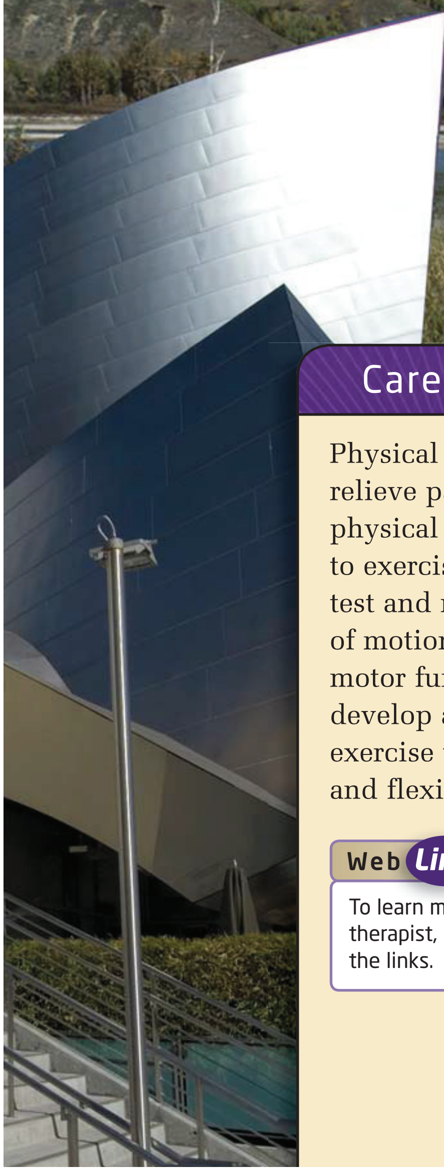
A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.

–Euclid, *The Elements*, Definition 8

Key Terms

- initial arm
- terminal arm
- angle in standard position
- reference angle
- exact value
- quadrantal angle
- sine law
- ambiguous case
- cosine law





Career Link

Physical therapists help improve mobility, relieve pain, and prevent or limit permanent physical disabilities by encouraging patients to exercise their muscles. Physical therapists test and measure the patient's strength, range of motion, balance, muscle performance, and motor functions. Next, physical therapists develop a treatment plan that often includes exercise to improve patient strength and flexibility.

Web Link

To learn more about the career of a physical therapist, go to www.mhrprecalc11.ca and follow the links.

2.1

Angles in Standard Position

Focus on...

- sketching an angle from 0° to 360° in standard position and determining its reference angle
- determining the quadrant in which an angle in standard position terminates
- determining the exact values of the sine, cosine, and tangent ratios of a given angle with reference angle 30°, 45°, or 60°
- solving problems involving trigonometric ratios

Do you think angles are only used in geometry? Angles occur in many everyday situations, such as driving: when you recline a car seat to a comfortable level, when you turn a wheel to ensure a safe turn around the corner, and when you angle a mirror to get the best view of vehicles behind you.

In architecture, angles are used to create more interesting and intriguing buildings. The use of angles in art is unlimited.

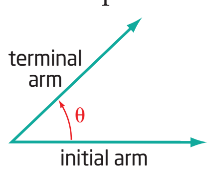
In sports, estimating angles is important in passing a hockey puck, shooting a basketball, and punting a football.

Look around you. How many angles can you identify in the objects in your classroom?

Jazz by Henri Matisse

Investigate Exact Values and Angles in Standard Position

In geometry, an angle is formed by two rays with a common endpoint. In trigonometry, angles are often interpreted as rotations of a ray. The starting position and the final position are called the initial arm and the terminal arm of the angle, respectively. If the angle of rotation is counterclockwise, then the angle is positive. In this chapter, all angles will be positive.



Part A: Angles in Standard Position

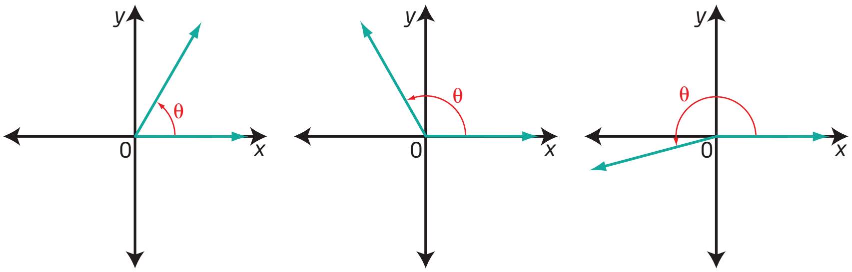
Materials

- grid paper
- ruler
- protractor

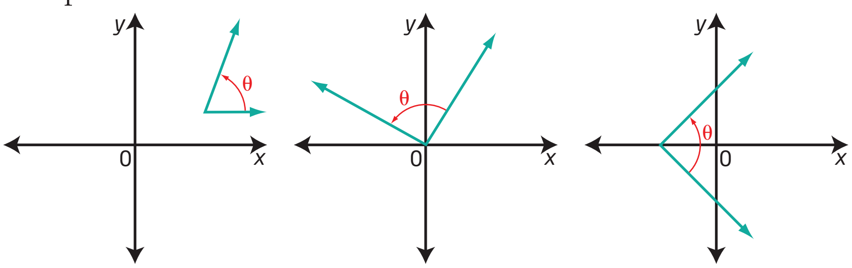
Work with a partner.

1. The diagrams in Group A show angles in standard position. The angles in Group B are not in standard position. How are the angles in Group A different from those in Group B? What characteristics do angles in standard position have?

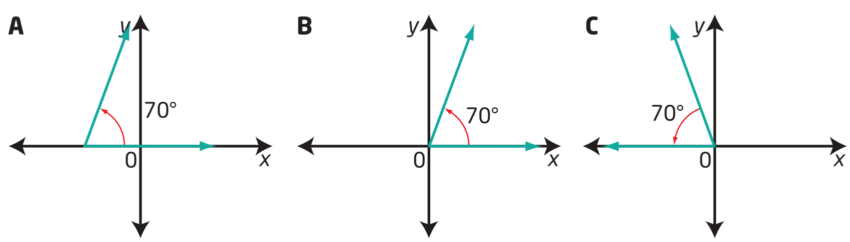
Group A:



Group B:



2. Which diagram shows an angle of 70° in standard position? Explain your choice.



3. On grid paper, draw coordinate axes. Then, use a protractor to draw angles in standard position with each of the following measures. Explain how you drew these angles. In which quadrant does the terminal arm of each angle lie?

a) 75° **b)** 105° **c)** 225° **d)** 320°

Reflect and Respond

4. Consider the angles that you have drawn. How might you define an angle in standard position?

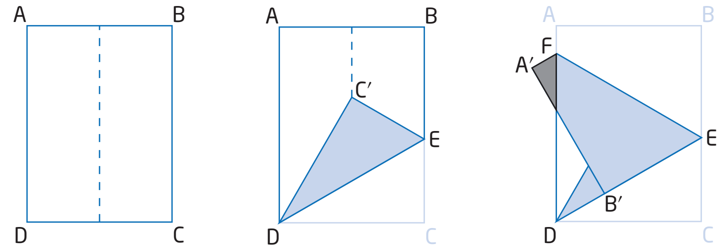
5. Explore and explain two ways to use a protractor to draw each angle in standard position.

a) 290° **b)** 200° **c)** 130° **d)** 325°

Part B: Create a 30°-60°-90° Triangle

6. Begin with an $8\frac{1}{2}'' \times 11''$ sheet of paper. Fold the paper in half lengthwise and make a crease down the middle.

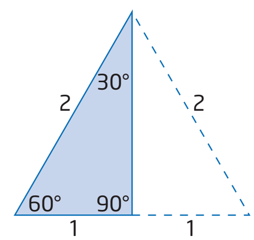
7. Unfold the paper. In Figure 1, the corners are labelled A, B, C, and D.



a) Take corner C to the centre fold line and make a crease, DE. See Figure 2.

b) Fold corner B so that BE lies on the edge of segment DE. The fold will be along line segment C′E. Fold the overlap (the grey-shaded region) under to complete the equilateral triangle (△DEF). See Figure 3.

8. For this activity, assume that the equilateral triangle has side lengths of 2 units.



a) To obtain a 30°-60°-90° triangle, fold the triangle in half, as shown.

b) Label the angles in the triangle as 30°, 60°, and 90°.

c) Use the Pythagorean Theorem to determine the exact measure of the third side of the triangle.

Did You Know?

Numbers such as $\sqrt{5}$ are irrational and cannot be written as terminating or repeating decimals. A length of $\sqrt{5}$ cm is an exact measure.

9. a) Write exact values for sin 30°, cos 30°, and tan 30°.

b) Write exact values for sin 60°, cos 60°, and tan 60°.

c) Can you use this triangle to determine the sine, cosine, and tangent ratios of 90°? Explain.

10. a) On a full sheet of grid paper, draw a set of coordinate axes.

b) Place your 30°-60°-90° triangle on the grid so that the vertex of the 60° angle is at the origin and the 90° angle sits in quadrant I as a perpendicular on the *x*-axis. What angle in standard position is modelled?

11. a) Reflect your triangle in the *y*-axis. What angle in standard position is modelled?

b) Reflect your original triangle in the *x*-axis. What angle in standard position is modelled?

c) Reflect your original triangle in the *y*-axis and then in the *x*-axis. What angle in standard position is modelled?

12. Repeat steps 10 and 11 with the 30° angle at the origin.

Reflect and Respond

13. When the triangle was reflected in an axis, what method did you use to determine the angle in standard position? Would this work for any angle?

14. As the triangle is reflected in an axis, how do you think that the values of the sine, cosine, and tangent ratios might change? Explain.

15. a) Do all 30°-60°-90° triangles have the side relationship of $1:\sqrt{3}:2$? Explain why or why not.

b) Use a ruler to measure the side lengths of your 30°-60°-90° triangle. Do the side lengths follow the relationship $1:\sqrt{3}:2$? How do you know?

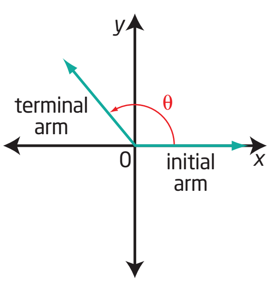
16. How can you create a 45°-45°-90° triangle by paper folding? What is the exact value of tan 45°? sin 45°? cos 45°?

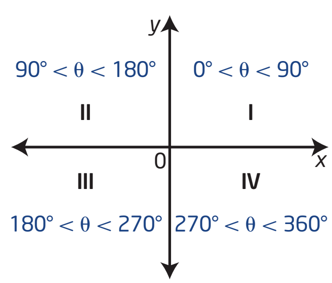
Link the Ideas

Angles in Standard Position, $0° \leq \theta < 360°$

On a Cartesian plane, you can generate an angle by rotating a ray about the origin. The starting position of the ray, along the positive x-axis, is the **initial arm** of the angle. The final position, after a rotation about the origin, is the **terminal arm** of the angle.

An angle is said to be an **angle in standard position** if its vertex is at the origin of a coordinate grid and its initial arm coincides with the positive x-axis.





Angles in standard position are always shown on the Cartesian plane. The x-axis and the y-axis divide the plane into four quadrants.

initial arm
- the arm of an angle in standard position that lies on the x-axis

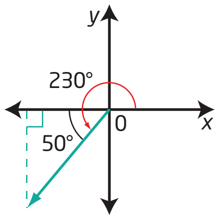
terminal arm
- the arm of an angle in standard position that meets the initial arm at the origin to form an angle

angle in standard position
- the position of an angle when its initial arm is on the positive x-axis and its vertex is at the origin

Reference Angles

reference angle

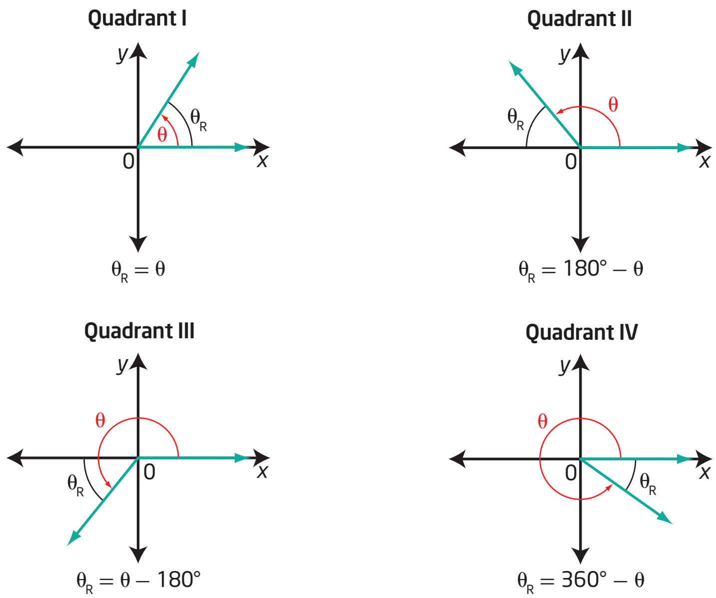
- the acute angle whose vertex is the origin and whose arms are the terminal arm of the angle and the *x*-axis



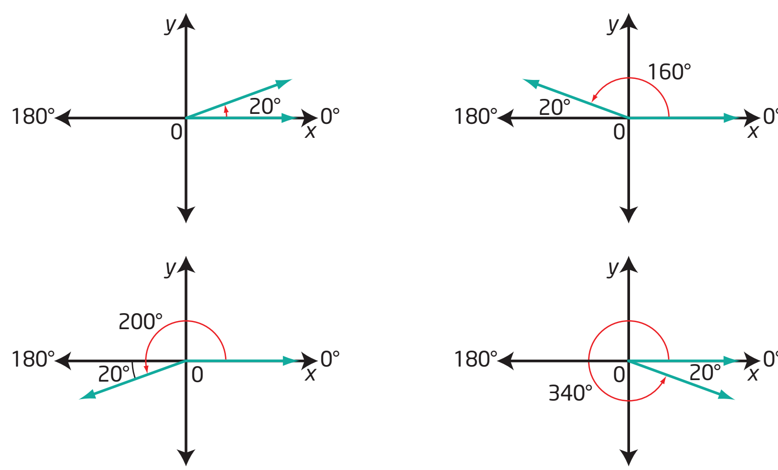
- the reference angle for 230° is 50°

For each angle in standard position, there is a corresponding acute angle called the **reference angle**. The reference angle is the acute angle formed between the terminal arm and the *x*-axis. The reference angle is always positive and measures between 0° and 90°. The trigonometric ratios of an angle in standard position are the same as the trigonometric ratios of its reference angle except that they may differ in sign. The right triangle that contains the reference angle and has one leg on the *x*-axis is known as the reference triangle.

The reference angle, θ_R, is illustrated for angles, θ, in standard position where $0° \leq \theta < 360°$.



The angles in standard position with a reference angle of 20° are 20°, 160°, 200°, and 340°.



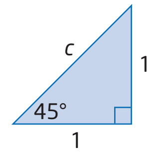
Special Right Triangles

For angles of 30°, 45°, and 60°, you can determine the **exact values** of trigonometric ratios.

exact value

- answers involving radicals are exact, unlike approximated decimal values
- fractions such as $\frac{1}{3}$ are exact, but an approximation of $\frac{1}{3}$ such as 0.333 is not

Drawing the diagonal of a square with a side length of 1 unit gives a 45°-45°-90° triangle. This is an isosceles right triangle.



Use the Pythagorean Theorem to find the length of the hypotenuse.

$$c^2 = a^2 + b^2$$
$$c^2 = 1^2 + 1^2$$
$$c^2 = 2$$
$$c = \sqrt{2}$$

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\sin 45° = \frac{1}{\sqrt{2}} \qquad \cos 45° = \frac{1}{\sqrt{2}} \qquad \tan 45° = \frac{1}{1}$$

$$\tan 45° = 1$$

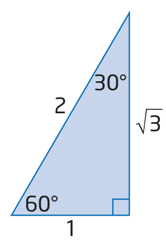
What are the three primary trigonometric ratios for the other acute angle in this triangle?

Drawing the altitude of an equilateral triangle with a side length of 2 units gives a 30°-60°-90° triangle.

Using the Pythagorean Theorem, the length of the altitude is $\sqrt{3}$ units.

$$\sin 60° = \frac{\sqrt{3}}{2} \qquad \cos 60° = \frac{1}{2} \qquad \tan 60° = \frac{\sqrt{3}}{1}$$
$$= \sqrt{3}$$

$$\sin 30° = \frac{1}{2} \qquad \cos 30° = \frac{\sqrt{3}}{2} \qquad \tan 30° = \frac{1}{\sqrt{3}}$$



Which trigonometric ratios for 30° have exact decimal values? Which are irrational numbers?

Example 1

Sketch an Angle in Standard Position, $0° \le \theta < 360°$

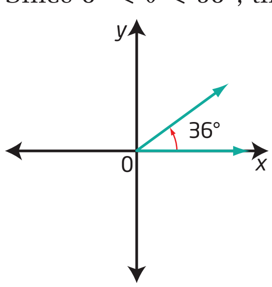
Sketch each angle in standard position. State the quadrant in which the terminal arm lies.

a) 36° **b)** 210° **c)** 315°

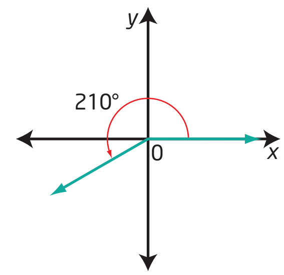
Solution

a) $\theta = 36°$

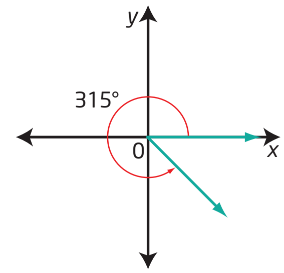
Since $0° < \theta < 90°$, the terminal arm of θ lies in quadrant I.



b) $\theta = 210°$
Since $180° < \theta < 270°$, the terminal arm of θ lies in quadrant III.



c) $\theta = 315°$
Since $270° < \theta < 360°$, the terminal arm of θ lies in quadrant IV.



Your Turn

Sketch each angle in standard position. State the quadrant in which the terminal arm lies.

a) 150° **b)** 60° **c)** 240°

Example 2

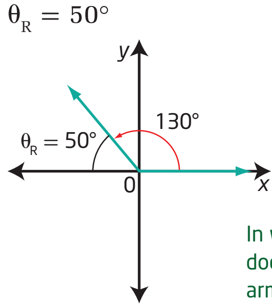
Determine a Reference Angle

Determine the reference angle θ_R for each angle θ. Sketch θ in standard position and label the reference angle θ_R.

a) $\theta = 130°$ **b)** $\theta = 300°$

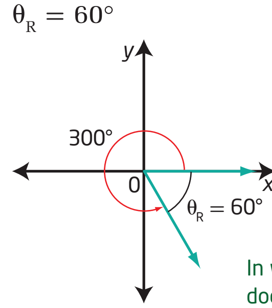
Solution

a) $\theta_R = 180° - 130°$
$\theta_R = 50°$



In which quadrant does the terminal arm of 130° lie?

b) $\theta_R = 360° - 300°$
$\theta_R = 60°$



In which quadrant does the terminal arm of 300° lie?

Your Turn

Determine the reference angle θ_R for each angle θ. Sketch θ and θ_R in standard position.

a) $\theta = 75°$ **b)** $\theta = 240°$

Example 3

Determine the Angle in Standard Position

Determine the angle in standard position when an angle of 40° is reflected

a) in the y-axis

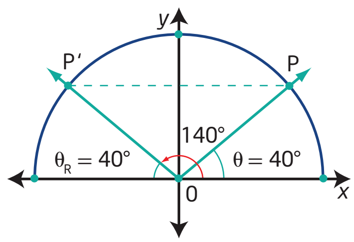
b) in the x-axis

c) in the y-axis and then in the x-axis

Solution

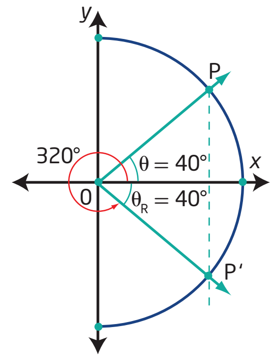
a) Reflecting an angle of 40° in the y-axis will result in a reference angle of 40° in quadrant II.

The measure of the angle in standard position for quadrant II is $180° - 40° = 140°$.



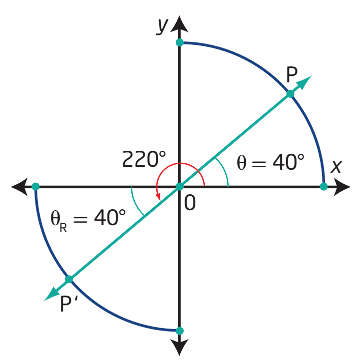
b) Reflecting an angle of 40° in the x-axis will result in a reference angle of 40° in quadrant IV.

The measure of the angle in standard position for quadrant IV is $360° - 40° = 320°$.



c) Reflecting an angle of 40° in the y-axis and then in the x-axis will result in a reference angle of 40° in quadrant III.

The measure of the angle in standard position for quadrant III is $180° + 40° = 220°$.



What angle of rotation of the original terminal arm would give the same terminal arm as this reflection?

Your Turn

Determine the angle in standard position when an angle of 60° is reflected

a) in the y-axis

b) in the x-axis

c) in the y-axis and then in the x-axis

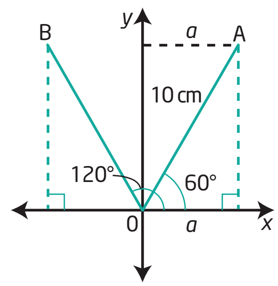
Example 4

Find an Exact Distance

Allie is learning to play the piano. Her teacher uses a metronome to help her keep time. The pendulum arm of the metronome is 10 cm long. For one particular tempo, the setting results in the arm moving back and forth from a start position of 60° to 120°. What horizontal distance does the tip of the arm move in one beat? Give an exact answer.

Solution

Draw a diagram to model the information.



OA represents the start position and OB the end position of the metronome arm for one beat. The tip of the arm moves a horizontal distance equal to a to reach the vertical position.

Find the horizontal distance a:

$$\cos 60° = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\frac{1}{2} = \frac{a}{10}$$

$$10\left(\frac{1}{2}\right) = a$$

$$5 = a$$

Why is $\frac{1}{2}$ substituted for cos 60°?

Because the reference angle for 120° is 60°, the tip moves the same horizontal distance past the vertical position to reach B.

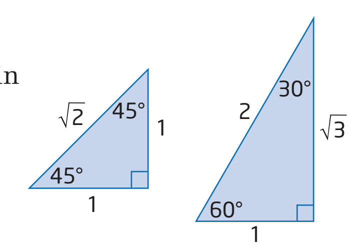
The exact horizontal distance travelled by the tip of the arm in one beat is 2(5) or 10 cm.

Your Turn

The tempo is adjusted so that the arm of the metronome swings from 45° to 135°. What exact horizontal distance does the tip of the arm travel in one beat?

Key Ideas

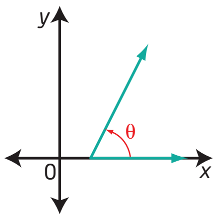
- An angle, θ, in standard position has its initial arm on the positive x-axis and its vertex at the origin. If the angle of rotation is counterclockwise, then the angle is positive.
- The reference angle is the acute angle whose vertex is the origin and whose arms are the x-axis and the terminal arm of θ.
- You can determine exact trigonometric ratios for angles of 30°, 45°, and 60° using special triangles.

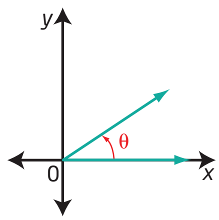


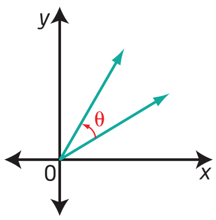
Check Your Understanding

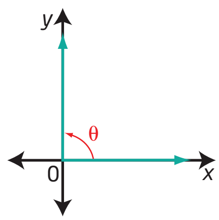
Practise

1. Is each angle, θ, in standard position? Explain.

a) 

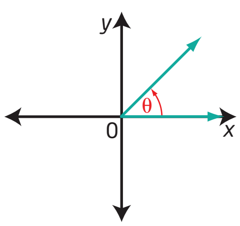
b) 

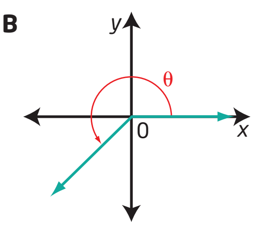
c) 

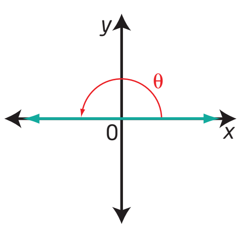
d) 

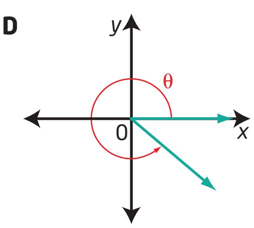
2. Without measuring, match each angle with a diagram of the angle in standard position.

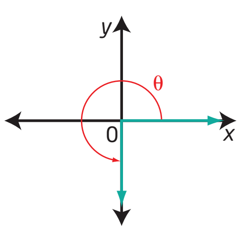
a) 150° **b)** 180°
c) 45° **d)** 320°
e) 215° **f)** 270°

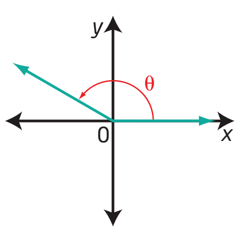
A 

B 

C 

D 

E 

F 

3. In which quadrant does the terminal arm of each angle in standard position lie?

a) 48° **b)** 300°
c) 185° **d)** 75°
e) 220° **f)** 160°

4. Sketch an angle in standard position with each given measure.

a) 70° **b)** 310°
c) 225° **d)** 165°

5. What is the reference angle for each angle in standard position?

a) 170° **b)** 345°
c) 72° **d)** 215°

6. Determine the measure of the three other angles in standard position, $0° < \theta < 360°$, that have a reference angle of

a) 45° **b)** 60°
c) 30° **d)** 75°

7. Copy and complete the table. Determine the measure of each angle in standard position given its reference angle and the quadrant in which the terminal arm lies.

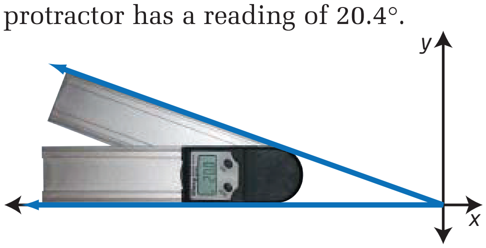
	Reference Angle	Quadrant	Angle in Standard Position
a)	72°	IV	
b)	56°	II	
c)	18°	III	
d)	35°	IV	

8. Copy and complete the table without using a calculator. Express each ratio using exact values.

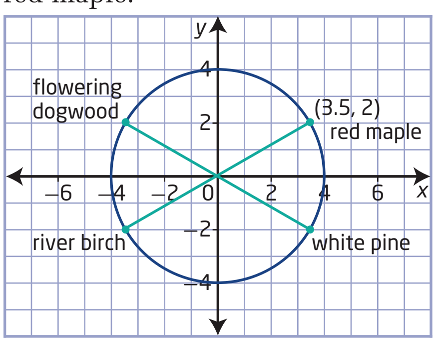
θ	sin θ	cos θ	tan θ
30°			
45°			
60°			

Apply

9. A digital protractor is used in woodworking. State the measure of the angle in standard position when the protractor has a reading of 20.4°.



10. Paul and Gail decide to use a Cartesian plane to design a landscape plan for their yard. Each grid mark represents a distance of 10 m. Their home is centred at the origin. There is a red maple tree at the point (3.5, 2). They will plant a flowering dogwood at a point that is a reflection in the *y*-axis of the position of the red maple. A white pine will be planted so that it is a reflection in the *x*-axis of the position of the red maple. A river birch will be planted so that it is a reflection in both the *x*-axis and the *y*-axis of the position of the red maple.



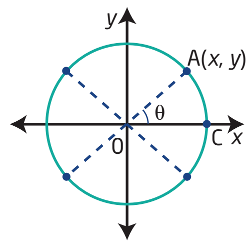
a) Determine the coordinates of the trees that Paul and Gail wish to plant.

b) Determine the angles in standard position if the lines drawn from the house to each of the trees are terminal arms. Express your answers to the nearest degree.

c) What is the actual distance between the red maple and the white pine?

11. A windshield wiper has a length of 50 cm. The wiper rotates from its resting position at 30°, in standard position, to 150°. Determine the exact horizontal distance that the tip of the wiper travels in one swipe.

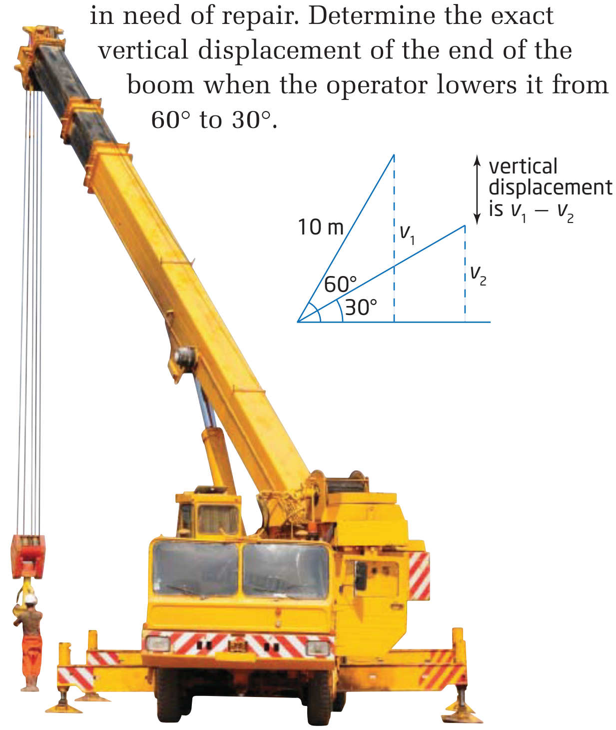
12. Suppose A(x, y) is a point on the terminal arm of ∠AOC in standard position.



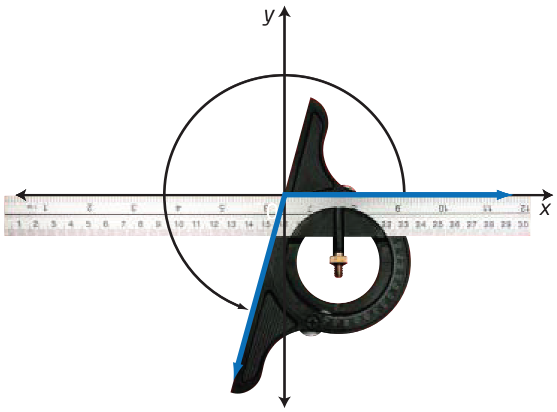
a) Determine the coordinates of points A′, A″, and A‴, where
   - A′ is the image of A reflected in the *x*-axis
   - A″ is the image of A reflected in the *y*-axis
   - A‴ is the image of A reflected in both the *x*-axis and the *y*-axis

b) Assume that each angle is in standard position and ∠AOC = θ. What are the measures, in terms of θ, of the angles that have A′, A″, and A‴ on their terminal arms?

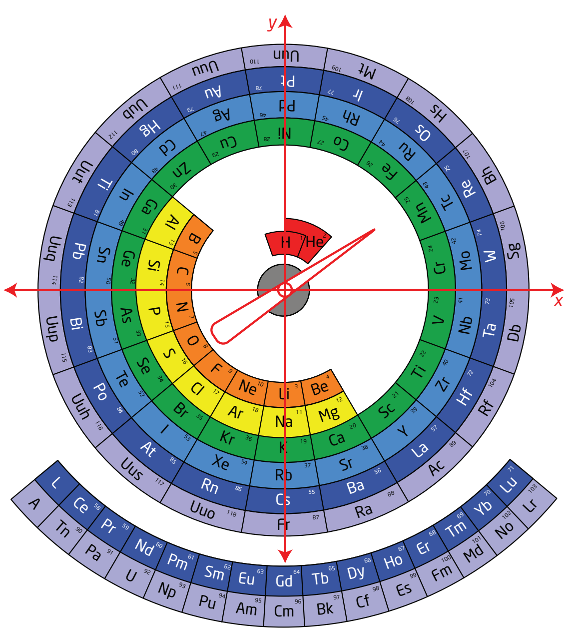
13. A 10-m boom lifts material onto a roof in need of repair. Determine the exact vertical displacement of the end of the boom when the operator lowers it from 60° to 30°.



14. Engineers use a bevel protractor to measure the angle and the depth of holes, slots, and other internal features. A bevel protractor is set to measure an angle of 72°. What is the measure of the angle in standard position of the lower half of the ruler, used for measuring the depth of an object?

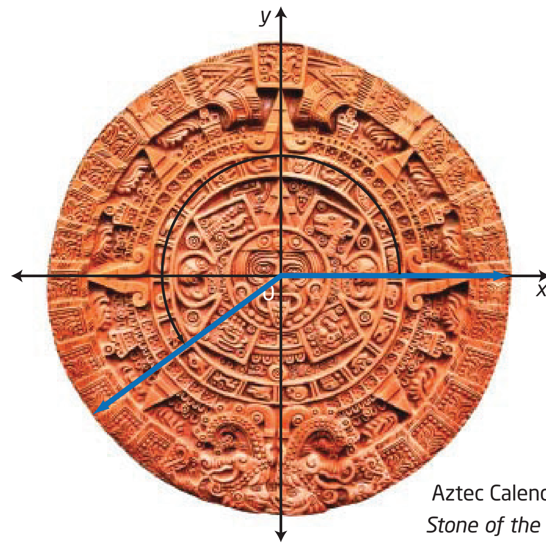


15. Researcher Mohd Abubakr developed a circular periodic table. He claims that his model gives a better idea of the size of the elements. Joshua and Andrea decided to make a spinner for the circular periodic table to help them study the elements for a quiz. They will spin the arm and then name the elements that the spinner lands on. Suppose the spinner lands so that it forms an angle in standard position of 110°. Name one of the elements it may have landed on.



16. The Aztec people of pre-Columbian Mexico used the Aztec Calendar. It consisted of a 365-day calendar cycle and a 260-day ritual cycle. In the stone carving of the calendar, the second ring from the centre showed the days of the month, numbered from one to 20.

Suppose the Aztec Calendar was placed on a Cartesian plane, as shown.



Aztec Calendar–
Stone of the Sun

a) The blue angle marks the passing of 12 days. Determine the measure of the angle.

b) How many days would have passed if the angle had been drawn in quadrant II, using the same reference angle as in part a)?

c) Keeping the same reference angle, how many days would have passed if the angle had been drawn in quadrant IV?

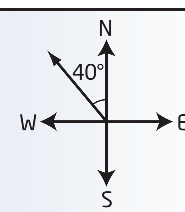
17. Express each direction as an angle in standard position. Sketch each angle.

a) N20°E **b)** S50°W

c) N80°W **d)** S15°E

Did You Know?

Directions are defined as a measure either east or west from north and south, measured in degrees. N40°W means to start from north and measure 40° toward the west.



Extend

18. You can use trigonometric ratios to design robotic arms. A robotic arm is motorized so that the angle, θ, increases at a constant rate of 10° per second from an initial angle of 0°. The arm is kept at a constant length of 45 cm to the tip of the fingers.

a) Let h represent the height of the robotic arm, measured at its fingertips. When $\theta = 0°$, h is 12 cm. Construct a table, using increments of 15°, that lists the angle, θ, and the height, h, for $0° \leq \theta \leq 90°$.

b) Does a constant increase in the angle produce a constant increase in the height? Justify your answer.

c) What conjecture would you make if θ were extended beyond 90°?

Did You Know?

A conjecture is a general conclusion based on a number of individual facts or results. In 1997, the American Mathematical Society published Beal's Conjecture. It states: If $A^x + B^y = C^z$, where A, B, C, x, y, and z are positive integers and x, y, and z are greater than 2, then A, B, and C must have a common prime factor. Andy Beal has offered a prize for a proof or counterexample of his conjecture.

Web Link

To learn about Beal's Conjecture and prize, go to www.mhrprecalc11.ca and follow the links.

19. Suppose two angles in standard position are supplementary and have terminal arms that are perpendicular. What are the measures of the angles?

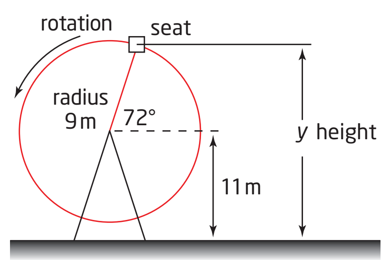
20. Carl and a friend are on the Antique Ferris Wheel Ride at Calaway Park in Calgary. The ride stops to unload the riders. Carl's seat forms an angle of 72° with a horizontal axis running through the centre of the Ferris wheel.

a) If the radius of the Ferris wheel is 9 m and the centre of the wheel is 11 m above the ground, determine the height of Carl's seat above the ground.

b) Suppose the Ferris wheel travels at four revolutions per minute and the operator stops the ride in 5 s.

i) Determine the angle in standard position of the seat that Carl is on at this second stop. Consider the horizontal central axis to be the x-axis.

ii) Determine the height of Carl's seat at the second stop.



Did You Know?

The first Ferris wheel was built for the 1853 World's Fair in Chicago. The wheel was designed by George Washington Gale Ferris. It had 36 gondola seats and reached a height of 80 m.

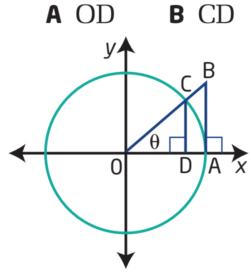
21. An angle in standard position is shown. Suppose the radius of the circle is 1 unit.

a) Which distance represents sin θ?

A OD **B** CD **C** OC **D** BA

b) Which distance represents tan θ?

A OD **B** CD **C** OC **D** BA



Create Connections

22. A point P(x, y) lies on the terminal arm of an angle θ. The distance from P to the origin is r units. Create a formula that links x, y, and r.

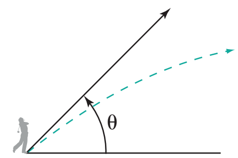
23. a) Copy and complete the table. Use a calculator and round ratios to four decimal places.

θ	20°	40°	60°	80°
sin θ				
sin (180° − θ)				
sin (180° + θ)				
sin (360° − θ)				

b) Make a conjecture about the relationships between sin θ, sin (180° − θ), sin (180° + θ), and sin (360° − θ).

c) Would your conjecture hold true for values of cosine and tangent? Explain your reasoning.

24. Daria purchased a new golf club. She wants to know the distance that she will be able to hit the ball with this club. She recalls from her physics class that the distance, d, a ball travels can be modelled by the formula $d = \frac{V^2 \cos \theta \sin \theta}{16}$, where V is the initial velocity, in feet per second, and θ is the angle of elevation.



a) The radar unit at the practice range indicates that the initial velocity is 110 ft/s and that the ball is hit at an angle of 30° to the ground. Determine the exact distance that Daria hit the ball with this driver.

b) To get a longer hit than that in part a), should Daria increase or decrease the angle of the hit? Explain.

c) What angle of elevation do you think would produce a hit that travels the greatest distance? Explain your reasoning.

Project Corner — Prospecting

- Prospecting is exploring an area for natural resources, such as oil, gas, minerals, precious metals, and mineral specimens. Prospectors travel through the countryside, often through creek beds and along ridgelines and hilltops, in search of natural resources.

Web Link

To search for locations of various minerals in Canada, go to www.mhrprecalc11.ca and follow the links.

2.2

Trigonometric Ratios of Any Angle

Focus on…

- determining the distance from the origin to a point (x, y) on the terminal arm of an angle
- determining the value of $\sin \theta$, $\cos \theta$, or $\tan \theta$ given any point (x, y) on the terminal arm of angle θ
- determining the value of $\sin \theta$, $\cos \theta$, or $\tan \theta$ for $\theta = 0°, 90°, 180°, 270°$, or $360°$
- solving for all values of θ in an equation involving sine, cosine, and tangent
- solving a problem involving trigonometric ratios

The Athabasca Oil Sands are located 40 km north of Fort McMurray, AB. They are the world's largest source of synthetic crude from oil sands, and the greatest single source in Canada. Since the beginning of the first oil sands production in 1967, technological advances have allowed for a tremendous increase in production and safety.

Massive machinery has been developed specifically for the excavation of the oil sands. Power shovels are equipped with a global positioning system (GPS) to make digging more exact. The operator must understand the angles necessary to operate the massive shovel. The design of power shovels uses the laws of trigonometry.

Did You Know?

Many Canadian companies are very aware of and sensitive to concerns about the impact of mining on the environment. The companies consult with local Aboriginal people on issues such as the re-establishment of native tree species, like low-bush cranberry and buffalo berry.

Investigate Trigonometric Ratios for Angles Greater Than 90°

Materials

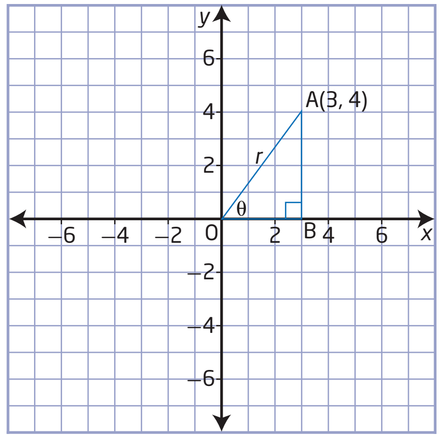
- grid paper
- protractor

1. On grid paper, draw a set of coordinate axes.

a) Plot the point A(3, 4). In which quadrant does the point A lie?

b) Draw the angle in standard position with terminal arm passing through point A.

2. Draw a line perpendicular to the x-axis through point A. Label the intersection of this line and the x-axis as point B. This point is on the initial arm of ∠AOB.



a) Use the Pythagorean Theorem to determine the length of the hypotenuse, r.

b) Write the primary trigonometric ratios for θ.

c) Determine the measure of θ, to the nearest degree.

3. How is each primary trigonometric ratio related to the coordinates of point A and the radius r?

4. a) Reflect point A in the y-axis to obtain point C. Draw a line segment from point C to the origin. What are the coordinates of point C?

b) Draw a line perpendicular to the x-axis through point C to create the reference triangle. Label the intersection of this line and the x-axis as point D. Use your answers from step 3 to write the primary trigonometric ratios for ∠COB.

5. a) What is the measure of ∠COB, to the nearest degree?

b) How are ∠COD and ∠COB related?

Reflect and Respond

6. a) Compare the trigonometric ratios for ∠AOB and ∠COB. What are the similarities and what are the differences?

b) Explain why some trigonometric ratios are positive and some are negative.

7. a) Reflect point C in the x-axis to obtain point E. Which trigonometric ratios would you expect to be positive? Which ones would you expect to be negative? Explain your reasoning.

b) Use the coordinates of point E and your definitions from step 3 to confirm your prediction.

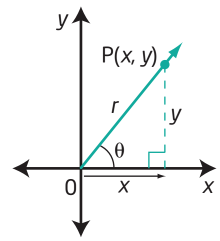
c) Extend this investigation into quadrant IV.

8. Make a table showing the signs of the sine, cosine, and tangent ratios of an angle, θ, in each of the four quadrants. Do you notice a pattern? How could you recognize the sign (positive or negative) of the trigonometric ratios in the various quadrants?

Link the Ideas

Finding the Trigonometric Ratios of Any Angle θ, where $0° \leq \theta < 360°$

Suppose θ is any angle in standard position, and P(x, y) is any point on its terminal arm, at a distance r from the origin. Then, by the Pythagorean Theorem, $r = \sqrt{x^2 + y^2}$.



You can use a reference triangle to determine the three primary trigonometric ratios in terms of x, y, and r.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

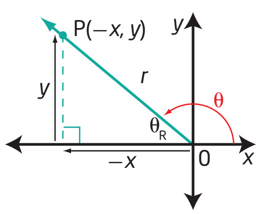
$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$

The chart below summarizes the signs of the trigonometric ratios in each quadrant. In each, the horizontal and vertical lengths are considered as directed distances.

Quadrant II
$90° < \theta < 180°$

$\sin\theta = \frac{y}{r}$ $\quad \cos\theta = \frac{-x}{r}$ $\quad \tan\theta = \frac{y}{-x}$

$\sin\theta > 0$ $\quad \cos\theta < 0$ $\quad \tan\theta < 0$

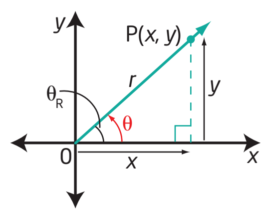


$\theta = 180° - \theta_R$

Quadrant I
$0° < \theta < 90°$

$\sin\theta = \frac{y}{r}$ $\quad \cos\theta = \frac{x}{r}$ $\quad \tan\theta = \frac{y}{x}$

$\sin\theta > 0$ $\quad \cos\theta > 0$ $\quad \tan\theta > 0$



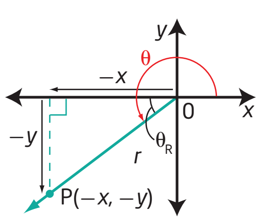
$\theta = \theta_R$

Why is r always positive?

Quadrant III
$180° < \theta < 270°$

$\sin\theta = \frac{-y}{r}$ $\quad \cos\theta = \frac{-x}{r}$ $\quad \tan\theta = \frac{-y}{-x}$

$\sin\theta < 0$ $\quad \cos\theta < 0$ $\quad \tan\theta > 0$

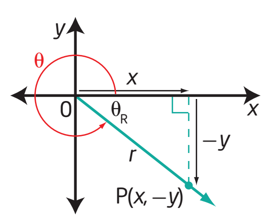


$\theta = 180° + \theta_R$

Quadrant IV
$270° < \theta < 360°$

$\sin\theta = \frac{-y}{r}$ $\quad \cos\theta = \frac{x}{r}$ $\quad \tan\theta = \frac{-y}{x}$

$\sin\theta < 0$ $\quad \cos\theta > 0$ $\quad \tan\theta < 0$



$\theta = 360° - \theta_R$

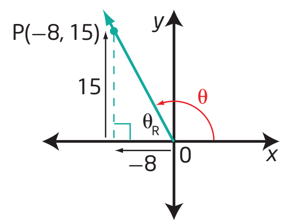
Example 1

Write Trigonometric Ratios for Angles in Any Quadrant

The point P(−8, 15) lies on the terminal arm of an angle, θ, in standard position. Determine the exact trigonometric ratios for sin θ, cos θ, and tan θ.

Solution

Sketch the reference triangle by drawing a line perpendicular to the *x*-axis through the point (−8, 15). The point P(−8, 15) is in quadrant II, so the terminal arm is in quadrant II.



Use the Pythagorean Theorem to determine the distance, *r*, from P(−8, 15) to the origin, (0, 0).

$r = \sqrt{x^2 + y^2}$

$r = \sqrt{(-8)^2 + (15)^2}$

$r = \sqrt{289}$

$r = 17$

The trigonometric ratios for θ can be written as follows:

$\sin\theta = \frac{y}{r}$	$\cos\theta = \frac{x}{r}$	$\tan\theta = \frac{y}{x}$
$\sin\theta = \frac{15}{17}$	$\cos\theta = \frac{-8}{17}$	$\tan\theta = \frac{15}{-8}$
	$\cos\theta = -\frac{8}{17}$	$\tan\theta = -\frac{15}{8}$

Your Turn

The point P(−5, −12) lies on the terminal arm of an angle, θ, in standard position. Determine the exact trigonometric ratios for sin θ, cos θ, and tan θ.

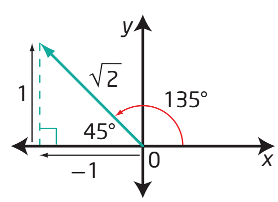
Example 2

Determine the Exact Value of a Trigonometric Ratio

Determine the exact value of cos 135°.

Solution

The terminal arm of 135° lies in quadrant II. The reference angle is 180° − 135°, or 45°. The cosine ratio is negative in quadrant II.



$\cos 135° = -\frac{1}{\sqrt{2}}$

Why are side lengths 1, 1, and $\sqrt{2}$ used?

Your Turn

Determine the exact value of sin 240°.

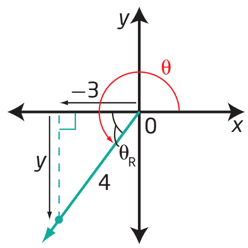
Example 3

Determine Trigonometric Ratios

Suppose θ is an angle in standard position with terminal arm in quadrant III, and $\cos\theta = -\frac{3}{4}$. What are the exact values of sin θ and tan θ?

Solution

Sketch a diagram.



Use the definition of cosine to find the exact values of x and r.

$$\cos\theta = \frac{x}{r}$$
$$\cos\theta = -\frac{3}{4}$$

Since the terminal arm is in quadrant III, x is negative. r is always positive. So, $x = -3$ and $r = 4$.

Use $x = -3$, $r = 4$ and the Pythagorean Theorem to find y.

$$x^2 + y^2 = r^2$$
$$(-3)^2 + y^2 = 4^2$$
$$9 + y^2 = 16$$
$$y^2 = 16 - 9$$
$$y^2 = 7$$
$$y = \pm\sqrt{7}$$

$y = \sqrt{7}$ is a solution for $y^2 = 7$ because $(\sqrt{7})(\sqrt{7}) = 7$
$y = -\sqrt{7}$ is also a solution because $(-\sqrt{7})(-\sqrt{7}) = 7$

Use $x = -3$, $y = -\sqrt{7}$, and $r = 4$ to write sin θ and tan θ.

Why is $-\sqrt{7}$ used for y here?

$$\sin\theta = \frac{y}{r} \qquad \tan\theta = \frac{y}{x}$$
$$\sin\theta = \frac{-\sqrt{7}}{4} \qquad \tan\theta = \frac{-\sqrt{7}}{-3}$$
$$\sin\theta = -\frac{\sqrt{7}}{4} \qquad \tan\theta = \frac{\sqrt{7}}{3}$$

Your Turn

Suppose θ is an angle in standard position with terminal arm in quadrant III, and $\tan\theta = \frac{1}{5}$. Determine the exact values of sin θ and cos θ.

Example 4

Determine Trigonometric Ratios of Quadrantal Angles

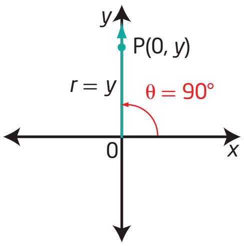
Determine the values of sin θ, cos θ, and tan θ when the terminal arm of **quadrantal angle** θ coincides with the positive y-axis, $\theta = 90°$.

quadrantal angle

- an angle in standard position whose terminal arm lies on one of the axes
- examples are 0°, 90°, 180°, 270°, and 360°

Solution

Let P(x, y) be any point on the positive y-axis. Then, $x = 0$ and $r = y$.



The trigonometric ratios can be written as follows.

$\sin 90° = \frac{y}{r}$ $\quad\cos 90° = \frac{x}{r}$ $\quad\tan 90° = \frac{y}{x}$

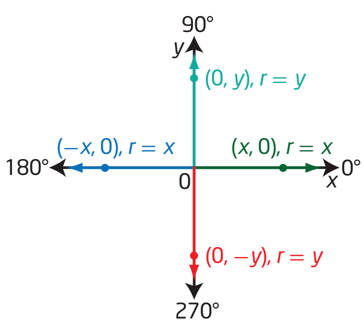
$\sin 90° = \frac{y}{y}$ $\quad\cos 90° = \frac{0}{y}$ $\quad\tan 90° = \frac{y}{0}$

$\sin 90° = 1$ $\quad\cos 90° = 0$ $\quad\tan 90°$ is undefined

Why is tan 90° undefined?

Your Turn

Use the diagram to determine the values of sin θ, cos θ, and tan θ for quadrantal angles of 0°, 180°, and 270°. Organize your answers in a table as shown below.



	0°	90°	180°	270°
sin θ		1		
cos θ		0		
tan θ		undefined		

Solving for Angles Given Their Sine, Cosine, or Tangent

Step 1 Determine which quadrants the solution(s) will be in by looking at the sign (+ or −) of the given ratio.

Step 2 Solve for the reference angle.

Why are the trigonometric ratios for the reference angle always positive?

Step 3 Sketch the reference angle in the appropriate quadrant. Use the diagram to determine the measure of the related angle in standard position.

Example 5

Solve for an Angle Given Its Exact Sine, Cosine, or Tangent Value

Solve for θ.

a) $\sin \theta = 0.5,\ 0° \le \theta < 360°$

b) $\cos \theta = -\frac{\sqrt{3}}{2},\ 0° \le \theta < 180°$

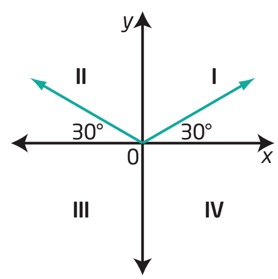
Solution

a) Since the ratio for sin θ is positive, the terminal arm lies in either quadrant I or quadrant II.

$\sin \theta_R = 0.5$

$\theta_R = 30°$

How do you know $\theta_R = 30°$?



In quadrant I, θ = 30°.

In quadrant II, $\theta = 180° - 30°$

$\theta = 150°$

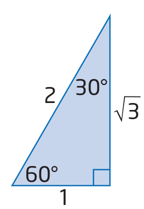
The solution to the equation sin θ = 0.5, $0 \le \theta < 360°$, is θ = 30° or θ = 150°.

b) Since the cosine ratio is negative, the terminal arm must lie in quadrant II or quadrant III. Given the restriction $0° \le \theta < 180°$, the terminal arm must lie in quadrant II.

Use a 30°-60°-90° triangle to determine the reference angle, θ_R.

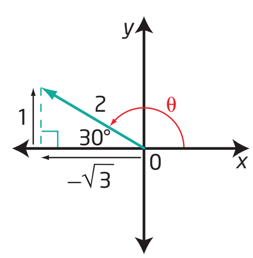
$\cos \theta_R = \frac{\sqrt{3}}{2}$

$\theta_R = 30°$



Using the reference angle of 30° in quadrant II, the measure of θ is 180° − 30° = 150°.

The solution to the equation $\cos \theta = -\frac{\sqrt{3}}{2}$, $0 \le \theta < 180°$, is θ = 150°.



Your Turn

Solve $\sin \theta = -\frac{1}{\sqrt{2}},\ 0° \le \theta < 360°$.

Example 6

Solve for an Angle Given Its Approximate Sine, Cosine, or Tangent Value

Given $\cos \theta = -0.6753$, where $0° \le \theta < 360°$, determine the measure of θ, to the nearest tenth of a degree.

Solution

The cosine ratio is negative, so the angles in standard position lie in quadrant II and quadrant III.

Use a calculator to determine the angle that has $\cos \theta_R = 0.6753$.

$\theta_R = \cos^{-1}(0.6753)$ — Why is $\cos^{-1}(0.6753)$ the reference angle?

$\theta_R \approx 47.5°$

With a reference angle of 47.5°, the measures of θ are as follows:

In quadrant II:
$\theta = 180° - 47.5°$
$\theta = 132.5°$

In quadrant III:
$\theta = 180° + 47.5°$
$\theta = 227.5°$

Your Turn

Determine the measure of θ, to the nearest degree, given $\sin \theta = -0.8090$, where $0° \le \theta < 360°$.

Key Ideas

- The primary trigonometric ratios for an angle, θ, in standard position that has a point P(x, y) on its terminal arm are $\sin \theta = \frac{y}{r}$, $\cos \theta = \frac{x}{r}$, and $\tan \theta = \frac{y}{x}$, where $r = \sqrt{x^2 + y^2}$.
- The table show the signs of the primary trigonometric ratios for an angle, θ, in standard position with the terminal arm in the given quadrant.

	Quadrant			
Ratio	**I**	**II**	**III**	**IV**
$\sin \theta$	+	+	−	−
$\cos \theta$	+	−	−	+
$\tan \theta$	+	−	+	−

- If the terminal arm of an angle, θ, in standard position lies on one of the axes, θ is called a quadrantal angle. The quadrantal angles are 0°, 90°, 180°, 270°, and 360°, $0° \le \theta \le 360°$.

Check Your Understanding

Practise

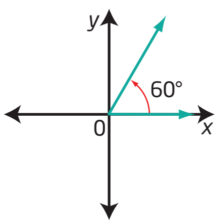
1. Sketch an angle in standard position so that the terminal arm passes through each point.

a) $(2, 6)$ **b)** $(-4, 2)$

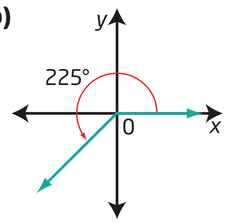
c) $(-5, -2)$ **d)** $(-1, 0)$

2. Determine the exact values of the sine, cosine, and tangent ratios for each angle.

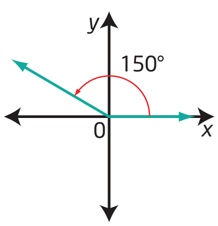
a)



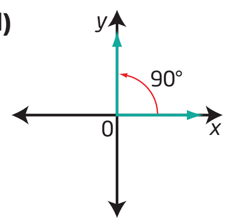
b)



c)

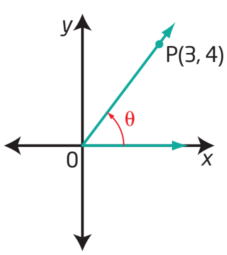


d)

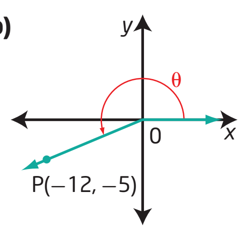


3. The coordinates of a point P on the terminal arm of each angle are shown. Write the exact trigonometric ratios sin θ, cos θ, and tan θ for each.

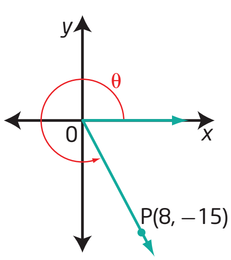
a)



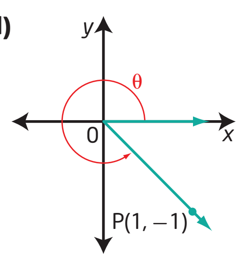
b)



c)



d)



4. For each description, in which quadrant does the terminal arm of angle θ lie?

a) $\cos \theta < 0$ and $\sin \theta > 0$

b) $\cos \theta > 0$ and $\tan \theta > 0$

c) $\sin \theta < 0$ and $\cos \theta < 0$

d) $\tan \theta < 0$ and $\cos \theta > 0$

5. Determine the exact values of sin θ, cos θ, and tan θ if the terminal arm of an angle in standard position passes through the given point.

a) P(−5, 12)

b) P(5, −3)

c) P(6, 3)

d) P(−24, −10)

6. Without using a calculator, state whether each ratio is positive or negative.

a) sin 155°

b) cos 320°

c) tan 120°

d) cos 220°

7. An angle is in standard position such that $\sin \theta = \frac{5}{13}$.

a) Sketch a diagram to show the two possible positions of the angle.

b) Determine the possible values of θ, to the nearest degree, if $0° \leq \theta < 360°$.

8. An angle in standard position has its terminal arm in the stated quadrant. Determine the exact values for the other two primary trigonometric ratios for each.

	Ratio Value	Quadrant
a)	$\cos \theta = -\frac{2}{3}$	II
b)	$\sin \theta = \frac{3}{5}$	I
c)	$\tan \theta = -\frac{4}{5}$	IV
d)	$\sin \theta = -\frac{1}{3}$	III
e)	$\tan \theta = 1$	III

9. Solve each equation, for $0° \le \theta < 360°$, using a diagram involving a special right triangle.

a) $\cos \theta = \frac{1}{2}$ **b)** $\cos \theta = -\frac{1}{\sqrt{2}}$

c) $\tan \theta = -\frac{1}{\sqrt{3}}$ **d)** $\sin \theta = -\frac{\sqrt{3}}{2}$

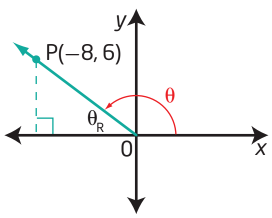
e) $\tan \theta = \sqrt{3}$ **f)** $\tan \theta = -1$

10. Copy and complete the table using the coordinates of a point on the terminal arm.

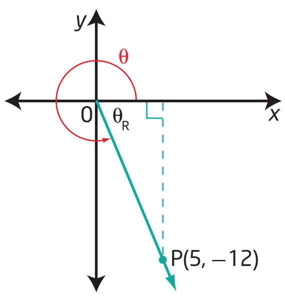
θ	sin θ	cos θ	tan θ
0°			
90°			
180°			
270°			
360°			

11. Determine the values of x, y, r, $\sin \theta$, $\cos \theta$, and $\tan \theta$ in each.

a)



b)



Apply

12. Point P(−9, 4) is on the terminal arm of an angle θ.

a) Sketch the angle in standard position.

b) What is the measure of the reference angle, to the nearest degree?

c) What is the measure of θ, to the nearest degree?

13. Point P(7, −24) is on the terminal arm of an angle, θ.

a) Sketch the angle in standard position.

b) What is the measure of the reference angle, to the nearest degree?

c) What is the measure of θ, to the nearest degree?

14. a) Determine sin θ when the terminal arm of an angle in standard position passes through the point P(2, 4).

b) Extend the terminal arm to include the point Q(4, 8). Determine sin θ for the angle in standard position whose terminal arm passes through point Q.

c) Extend the terminal arm to include the point R(8, 16). Determine sin θ for the angle in standard position whose terminal arm passes through point R.

d) Explain your results from parts a), b), and c). What do you notice? Why does this happen?

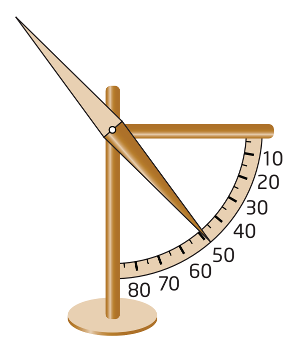
15. The point P(k, 24) is 25 units from the origin. If P lies on the terminal arm of an angle, θ, in standard position, $0° \le \theta < 360°$, determine

a) the measure(s) of θ

b) the sine, cosine, and tangent ratios for θ

16. If $\cos \theta = \frac{1}{5}$ and $\tan \theta = 2\sqrt{6}$, determine the exact value of sin θ.

17. The angle between the horizontal and Earth's magnetic field is called the angle of dip. Some migratory birds may be capable of detecting changes in the angle of dip, which helps them navigate. The angle of dip at the magnetic equator is 0°, while the angle at the North and South Poles is 90°. Determine the exact values of sin θ, cos θ, and tan θ for the angles of dip at the magnetic equator and the North and South Poles.



18. Without using technology, determine whether each statement is true or false. Justify your answer.

a) $\sin 151° = \sin 29°$

b) $\cos 135° = \sin 225°$

c) $\tan 135° = \tan 225°$

d) $\sin 60° = \cos 330°$

e) $\sin 270° = \cos 180°$

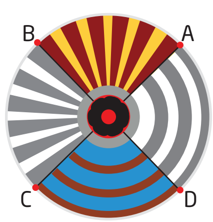
19. Copy and complete the table. Use exact values. Extend the table to include the primary trigonometric ratios for all angles in standard position, $90° \leq \theta \leq 360°$, that have the same reference angle as those listed for quadrant I.

θ	sin θ	cos θ	tan θ
0°			
30°			
45°			
60°			
90°			

20. Alberta Aboriginal Tourism designed a circular icon that represents both the Métis and First Nations communities of Alberta. The centre of the icon represents the collection of all peoples' perspectives and points of view relating to Aboriginal history, touching every quadrant and direction.

a) Suppose the icon is placed on a coordinate plane with a reference angle of 45° for points A, B, C, and D. Determine the measure of the angles in standard position for points A, B, C, and D.

b) If the radius of the circle is 1 unit, determine the coordinates of points A, B, C, and D.



21. Explore patterns in the sine, cosine, and tangent ratios.

a) Copy and complete the table started below. List the sine, cosine, and tangent ratios for θ in increments of 15° for $0° \leq \theta \leq 180°$. Where necessary, round values to four decimal places.

Angle	Sine	Cosine	Tangent
0°			
15°			
30°			
45°			
60°			

b) What do you observe about the sine, cosine, and tangent ratios as θ increases?

c) What comparisons can you make between the sine and cosine ratios?

d) Determine the signs of the ratios as you move from quadrant I to quadrant II.

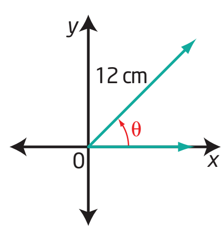
e) Describe what you expect will happen if you expand the table to include quadrant III and quadrant IV.

Extend

22. a) The line $y = 6x$, for $x \geq 0$, creates an acute angle, θ, with the x-axis. Determine the sine, cosine, and tangent ratios for θ.

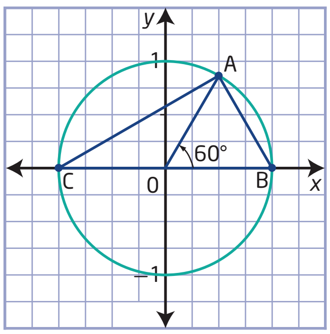
b) If the terminal arm of an angle, θ, lies on the line $4y + 3x = 0$, for $x \geq 0$, determine the exact value of $\tan \theta + \cos \theta$.

23. Consider an angle in standard position with $r = 12$ cm. Describe how the measures of x, y, $\sin \theta$, $\cos \theta$, and $\tan \theta$ change as θ increases continuously from 0° to 90°.



24. Suppose θ is a positive acute angle and $\cos \theta = a$. Write an expression for $\tan \theta$ in terms of a.

25. Consider an angle of 60° in standard position in a circle of radius 1 unit. Points A, B, and C lie on the circumference, as shown. Show that the lengths of the sides of △ABC satisfy the Pythagorean Theorem and that ∠CAB = 90°.



Create Connections

26. Explain how you can use reference angles to determine the trigonometric ratios of any angle, θ.

27. Point P(−5, −9) is on the terminal arm of an angle, θ, in standard position. Explain the role of the reference triangle and the reference angle in determining the value of θ.

28. Explain why there are exactly two non-quadrantal angles between 0° and 360° that have the same sine ratio.

29. Suppose that θ is an angle in standard position with $\cos \theta = -\frac{1}{2}$ and $\sin \theta = -\frac{\sqrt{3}}{2}$, $0° \le \theta < 360°$. Determine the measure of θ. Explain your reasoning, including diagrams.

30. **MINI LAB** Use dynamic geometry software to explore the trigonometric ratios.

Step 1 **a)** Draw a circle with a radius of 5 units and centre at the origin.

b) Plot a point A on the circle in quadrant I. Join point A and the origin by constructing a line segment. Label this distance r.

Step 2 **a)** Record the x-coordinate and the y-coordinate for point A.

b) Construct a formula to calculate the sine ratio of the angle in standard position whose terminal arm passes through point A. Use the measure and calculate features of your software to determine the sine ratio of this angle.

c) Repeat step b) to determine the cosine ratio and tangent ratio of the angle in standard position whose terminal arm passes through point A.

Step 3 Animate point A. Use the motion controller to slow the animation. Pause the animation to observe the ratios at points along the circle.

Step 4 **a)** What observations can you make about the sine, cosine, and tangent ratios as point A moves around the circle?

b) Record where the sine and cosine ratios are equal. What is the measure of the angle at these points?

c) What do you notice about the signs of the ratios as point A moves around the circle? Explain.

d) For several choices for point A, divide the sine ratio by the cosine ratio. What do you notice about this calculation? Is it true for all angles as A moves around the circle?

2.3

The Sine Law

Focus on…

- using the primary trigonometric ratios to solve problems involving triangles that are not right triangles
- recognizing when to use the sine law to solve a given problem
- sketching a diagram to represent a problem involving the sine law
- explaining a proof of the sine law
- solving problems using the sine law
- solving problems involving the ambiguous case of the sine law

How is an airplane pilot able to make precise landings even at night or in poor visibility? Airplanes have instrument landing systems that allow pilots to receive precise lateral and vertical guidance on approach and landing. Since 1994, airplanes have used the global positioning system (GPS) to provide the pilot with data on an approach. To understand the GPS, a pilot must understand the trigonometry of triangulation.

You can use right-triangle trigonometry to solve problems involving right triangles. However, many interesting problems involve oblique triangles. Oblique triangles are any triangles that do not contain a right angle. In this section, you will use right-triangle trigonometry to develop the sine law. You can use the sine law to solve some problems involving non-right triangles.

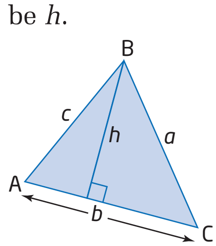
Investigate the Sine Law

Materials

- protractor

1. In an oblique triangle, the ratio of the sine of an angle to the length of its opposite side is constant. Demonstrate that this is true by drawing and measuring any oblique triangle. Compare your results with those of other students.

2. Draw an oblique triangle. Label its vertices A, B, and C and its side lengths a, b, and c. Draw an altitude from B to AC and let its height be h.



3. Use the two right triangles formed. Write a trigonometric ratio for sin A. Repeat for sin C. How are the two equations alike?
4. Rearrange each equation from step 3, expressing h in terms of the side and the sine of the angle.
5. a) Relate the two equations from step 4 to eliminate h and form one equation.
   b) Divide both sides of the equation by ac.

Reflect and Respond

6. The steps so far have led you to a partial equation for the sine law.
   a) Describe what measures in a triangle the sine law connects.
   b) What components do you need to be able to use the sine law?
7. Demonstrate how you could expand the ratios from step 5 to include the third ratio, $\frac{\sin B}{b}$.
8. Together, steps 5 and 7 form the sine law. Write out the sine law that you have derived and state it in words.
9. Can you solve all oblique triangles using the sine law? If not, give an example where the sine law does not allow you to solve for unknown angle(s) or side(s).

Link the Ideas

You have previously encountered problems involving right triangles that you could solve using the Pythagorean Theorem and the primary trigonometric ratios. However, a triangle that models a situation with unknown distances or angles may not be a right triangle. One method of solving an oblique triangle is to use the sine law. To prove the sine law, you need to extend your earlier skills with trigonometry.

Did You Know?

Nasir al-Din al-Tusi, born in the year 1201 C.E., began his career as an astronomer in Baghdad. In *On the Sector Figure*, he derived the sine law.

The Sine Law

sine law

- the sides of a triangle are proportional to the sines of the opposite angles
- $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

The **sine law** is a relationship between the sides and angles in any triangle. Let △ABC be any triangle, where a, b, and c represent the measures of the sides opposite ∠A, ∠B, and ∠C, respectively. Then,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

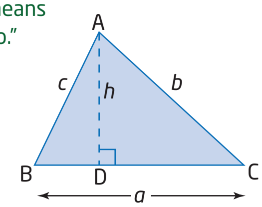
or

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Proof

In △ABC, draw an altitude AD ⊥ BC.
Let AD = h.

The symbol ⊥ means "perpendicular to."



In △ABD:

$$\sin B = \frac{h}{c}$$
$$h = c \sin B$$

In △ACD:

$$\sin C = \frac{h}{b}$$
$$h = b \sin C$$

Relate these two equations, because both equal h:

$c \sin B = b \sin C$ Divide both sides by sin B sin C.

$$\frac{c}{\sin C} = \frac{b}{\sin B}$$

This is part of the sine law.

By drawing the altitude from C and using similar steps, you can show that

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

Therefore,

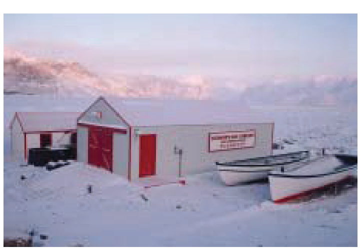
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

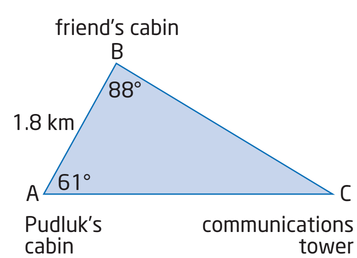
or

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Example 1

Determine an Unknown Side Length





Pudluk's family and his friend own cabins on the Kalit River in Nunavut. Pudluk and his friend wish to determine the distance from Pudluk's cabin to the store on the edge of town. They know that the distance between their cabins is 1.8 km. Using a transit, they estimate the measures of the angles between their cabins and the communications tower near the store, as shown in the diagram. Determine the distance from Pudluk's cabin to the store, to the nearest tenth of a kilometre.

Solution

Method 1: Use Primary Trigonometric Ratios

Calculate the measure of ∠C.

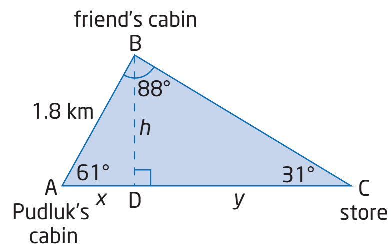
$\angle C = 180° - 88° - 61°$

$\angle C = 31°$

What relationship exists for the sum of the interior angles of any triangle?

Draw the altitude of the triangle from B to intersect AC at point D. Label the altitude *h*.

The distance from Pudluk's cabin to the store is the sum of the distances AD and DC.



From △ABD, determine *h*.

$$\sin 61° = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\sin 61° = \frac{h}{1.8}$$

$$h = 1.8 \sin 61°$$

Check that your calculator is in degree mode.

From △ABD, determine *x*.

$$\cos 61° = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\cos 61° = \frac{x}{1.8}$$

$$x = 1.8 \cos 61°$$

$$x = 0.872\ldots$$

From △BDC, determine *y*.

$$\tan 31° = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan 31° = \frac{h}{y}$$

$$y = \frac{1.8 \sin 61°}{\tan 31°}$$

$$y = 2.620\ldots$$

Then, AC = $x + y$, or 3.492….
The distance from Pudluk's cabin to the store is approximately 3.5 km.

Method 2: Use the Sine Law

Calculate the measure of ∠C.

$\angle C = 180° - 88° - 61°$

$\angle C = 31°$

What is the sum of the interior angles of any triangle?

List the measures.

$\angle A = 61°$ $a =$ ■

$\angle B = 88°$ $b =$ ■

$\angle C = 31°$ $c = 1.8$ km

Which pairs of ratios from the sine law would you use to solve for *b*?

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

Why is this form of the sine law used?

$$\frac{b}{\sin 88°} = \frac{1.8}{\sin 31°}$$

What do you do to each side to isolate *b*?

$$b = \frac{1.8 \sin 88°}{\sin 31°}$$

$$b = 3.492\ldots$$

Compare the two methods. Which do you prefer and why?

The distance from Pudluk's cabin to the store is approximately 3.5 km.

Your Turn

Determine the distance from Pudluk's friend's cabin to the store.

Example 2

Determine an Unknown Angle Measure

In $\triangle PQR$, $\angle P = 36°$, $p = 24.8$ m, and $q = 23.4$ m. Determine the measure of $\angle R$, to the nearest degree.

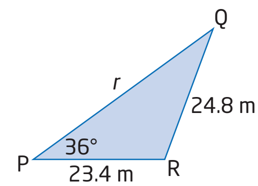
Solution

Sketch a diagram of the triangle. List the measures.

$\angle P = 36°$ $\quad p = 24.8$
$\angle Q = ■$ $\quad q = 23.4$
$\angle R = ■$ $\quad r = ■$

Since $p > q$, there is only one possible triangle.

Which ratios would you use?



Use the sine law to determine $\angle Q$.

Why do you need to determine ∠Q?

$$\frac{\sin Q}{q} = \frac{\sin P}{p}$$
$$\frac{\sin Q}{23.4} = \frac{\sin 36°}{24.8}$$
$$\sin Q = \frac{23.4 \sin 36°}{24.8}$$
$$\angle Q = \sin^{-1}\left(\frac{23.4 \sin 36°}{24.8}\right)$$
$$\angle Q = 33.68\ldots$$

Thus, $\angle Q$ is 34°, to the nearest degree.

Use the angle sum of a triangle to determine $\angle R$.

$\angle R = 180° - 34° - 36°$
$\angle R = 110°$

The measure of $\angle R$ is 110°, to the nearest degree.

Your Turn

In $\triangle LMN$, $\angle L = 64°$, $l = 25.2$ cm, and $m = 16.5$ cm. Determine the measure of $\angle N$, to the nearest degree.

The Ambiguous Case

When solving a triangle, you must analyse the given information to determine if a solution exists. If you are given the measures of two angles and one side (ASA), then the triangle is uniquely defined. However, if you are given two sides and an angle opposite one of those sides (SSA), the **ambiguous case** may occur. In the ambiguous case, there are three possible outcomes:

- no triangle exists that has the given measures; there is no solution
- one triangle exists that has the given measures; there is one solution
- two distinct triangles exist that have the given measures; there are two distinct solutions

ambiguous case

- from the given information the solution for the triangle is not clear: there might be one triangle, two triangles, or no triangle

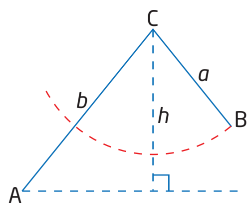
These possibilities are summarized in the diagrams below.

Suppose you are given the measures of side b and $\angle$A of $\triangle$ABC. You can find the height of the triangle by using $h = b \sin A$.

Why can you use this equation to find the height?

In $\triangle$ABC, $\angle$A and side b are constant because they are given. Consider different possible lengths of side a.

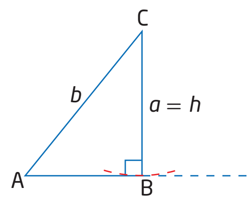
For an acute $\angle$A, the four possible lengths of side a result in four different cases.



$a < h$
$a < b \sin A$
no solution

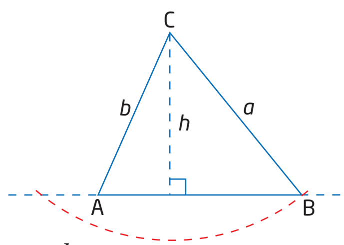
Why is there no solution in this case?

Recall that $h = b \sin A$.



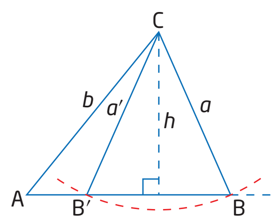
$a = h$
$a = b \sin A$
one solution

What type of triangle occurs in this case?



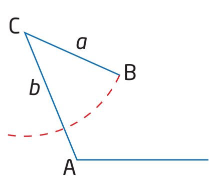
$a \geq b$
one solution

Why is there not another solution with B on the left side of A?

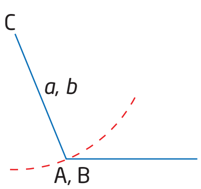


$h < a < b$
$b \sin A < a < b$
two solutions

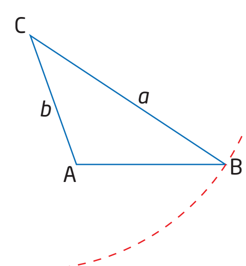
For an obtuse $\angle$A, three cases can occur.



$a < b$
no solution



$a = b$
no solution



$a > b$
one solution

Example 3

Use the Sine Law in an Ambiguous Case

In $\triangle ABC$, $\angle A = 30°$, $a = 24$ cm, and $b = 42$ cm. Determine the measures of the other side and angles. Round your answers to the nearest unit.

Solution

List the measures.

$\angle A = 30°$ $\quad a = 24$ cm
$\angle B = ■$ $\quad b = 42$ cm
$\angle C = ■$ $\quad c = ■$

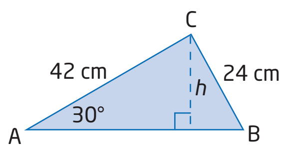
Because two sides and an angle opposite one of the sides are known, it is possible that this triangle may have two different solutions, one solution, or no solution. $\angle A$ is acute and $a < b$, so check which condition is true.

$a < b \sin A$: no solution
$a = b \sin A$: one solution
$a > b \sin A$: two solutions

Why is the value of $b \sin A$ so important?

Sketch a possible diagram.

Where does the length of CB actually fit?



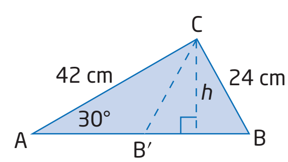
Determine the height of the triangle.

$$\sin A = \frac{h}{b}$$
$$h = b \sin A$$
$$h = 42 \sin 30°$$
$$h = 21$$

How do you know the value of sin 30°?

Since $24 > 21$, the case $a > b \sin A$ occurs.

Therefore, two triangles are possible. The second solution will give an obtuse angle for $\angle B$.



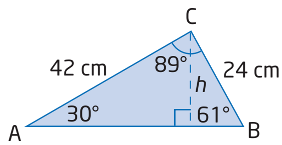
Solve for $\angle B$ using the sine law.

$$\frac{\sin B}{b} = \frac{\sin A}{a}$$
$$\frac{\sin B}{42} = \frac{\sin 30°}{24}$$
$$\sin B = \frac{42 \sin 30°}{24}$$
$$\angle B = \sin^{-1}\left(\frac{42 \sin 30°}{24}\right)$$
$$\angle B = 61.044\ldots$$

To the nearest degree, $\angle B = 61°$.

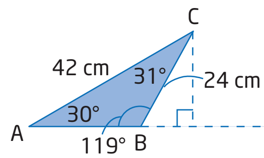
To find the second possible measure of $\angle B$, use $61°$ as the reference angle in quadrant II. Then, $\angle B = 180° - 61°$ or $\angle B = 119°$.

Case 1: ∠B = 61°
∠C = 180° − (61° + 30°)
∠C = 89°



$$\frac{c}{\sin 89°} = \frac{24}{\sin 30°}$$
$$c = \frac{24 \sin 89°}{\sin 30°}$$
$$c = 47.992\ldots$$

Case 2: ∠B = 119°
∠C = 180° − (119° + 30°)
∠C = 31°



$$\frac{c}{\sin 31°} = \frac{24}{\sin 30°}$$
$$c = \frac{24 \sin 31°}{\sin 30°}$$
$$c = 24.721\ldots$$

Use the sine law to determine the measure of side c in each case.

The two possible triangles are as follows:
acute △ABC: ∠A = 30°, ∠B = 61°, ∠C = 89°,
a = 24 cm, b = 42 cm, c = 48 cm
obtuse △ABC: ∠A = 30°, ∠B = 119°, ∠C = 31°,
a = 24 cm, b = 42 cm, c = 25 cm

Compare the ratios $\frac{a}{\sin A}$, $\frac{b}{\sin B}$, and $\frac{c}{\sin C}$ to check your answers.

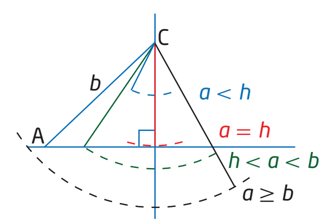
Your Turn

In △ABC, ∠A = 39°, a = 14 cm, and b = 10 cm. Determine the measures of the other side and angles. Express your answers to the nearest unit.

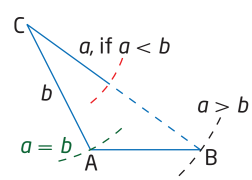
Key Ideas

- You can use the sine law to find the measures of sides and angles in a triangle.
- For △ABC, state the sine law as $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ or $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.
- Use the sine law to solve a triangle when you are given the measures of
  - two angles and one side
  - two sides and an angle that is opposite one of the given sides
- The ambiguous case of the sine law may occur when you are given two sides and an angle opposite one of the sides.
- For the ambiguous case in △ABC, when ∠A is an acute angle:
  - $a \geq b$ one solution
  - $a = h$ one solution
  - $a < h$ no solution
  - $b \sin A < a < b$ two solutions

$h = b \sin A$



- For the ambiguous case in △ABC, when ∠A is an obtuse angle:
  - $a \leq b$ no solution
  - $a > b$ one solution



Check Your Understanding

Practise

Where necessary, round lengths to the nearest tenth of a unit and angle measures to the nearest degree.

1. Solve for the unknown side or angle in each.

a) $\dfrac{a}{\sin 35°} = \dfrac{10}{\sin 40°}$

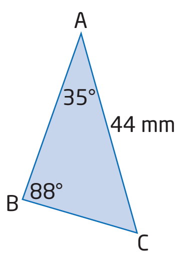
b) $\dfrac{b}{\sin 48°} = \dfrac{65}{\sin 75°}$

c) $\dfrac{\sin \theta}{12} = \dfrac{\sin 50°}{65}$

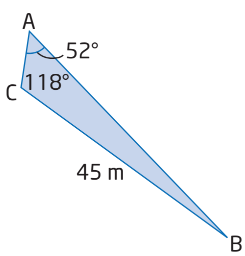
d) $\dfrac{\sin A}{25} = \dfrac{\sin 62°}{32}$

2. Determine the length of AB in each.

a)

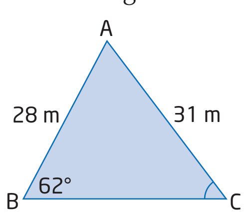


b)

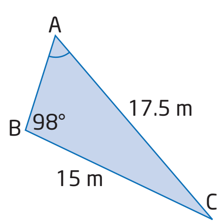


3. Determine the value of the marked unknown angle in each.

a)

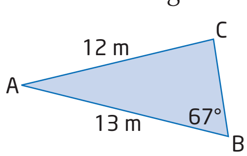


b)

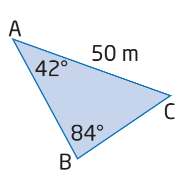


4. Determining the lengths of all three sides and the measures of all three angles is called solving a triangle. Solve each triangle.

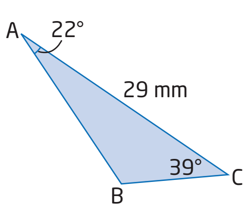
a)



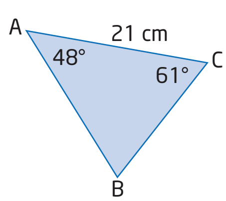
b)



c)



d)



5. Sketch each triangle. Determine the measure of the indicated side.

a) In △ABC, ∠A = 57°, ∠B = 73°, and AB = 24 cm. Find the length of AC.

b) In △ABC, ∠B = 38°, ∠C = 56°, and BC = 63 cm. Find the length of AB.

c) In △ABC, ∠A = 50°, ∠B = 50°, and AC = 27 m. Find the length of AB.

d) In △ABC, ∠A = 23°, ∠C = 78°, and AB = 15 cm. Find the length of BC.

6. For each triangle, determine whether there is no solution, one solution, or two solutions.

a) In △ABC, ∠A = 39°, a = 10 cm, and b = 14 cm.

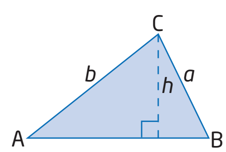
b) In △ABC, ∠A = 123°, a = 23 cm, and b = 12 cm.

c) In △ABC, ∠A = 145°, a = 18 cm, and b = 10 cm.

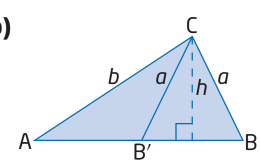
d) In △ABC, ∠A = 124°, a = 1 cm, and b = 2 cm.

7. In each diagram, h is an altitude. Describe how ∠A, sides a and b, and h are related in each diagram.

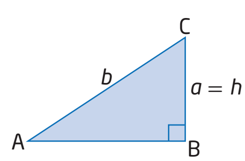
a)



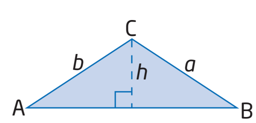
b)



c)



d)



8. Determine the unknown side and angles in each triangle. If two solutions are possible, give both.

a) In △ABC, ∠C = 31°, a = 5.6 cm, and c = 3.9 cm.

b) In △PQR, ∠Q = 43°, p = 20 cm, and q = 15 cm.

c) In △XYZ, ∠X = 53°, x = 8.5 cm, and z = 12.3 cm.

9. In △ABC, ∠A = 26° and b = 120 cm. Determine the range of values of a for which there is

a) one oblique triangle

b) one right triangle

c) two oblique triangles

d) no triangle

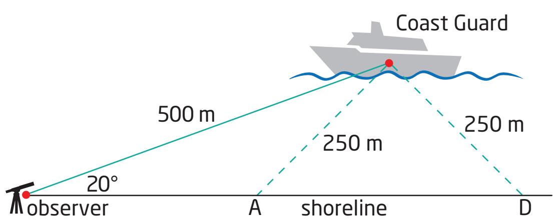
Apply

10. A hot-air balloon is flying above BC Place Stadium. Maria is standing due north of the stadium and can see the balloon at an angle of inclination of 64°. Roy is due south of the stadium and can see the balloon at an angle of inclination of 49°. The horizontal distance between Maria and Roy is 500 m.

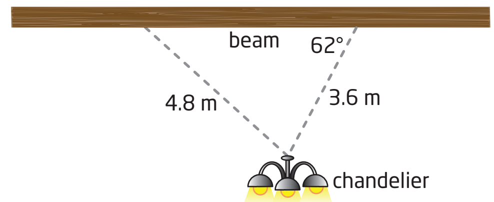
a) Sketch a diagram to represent the given information.

b) Determine the distance that the hot air balloon is from Maria.

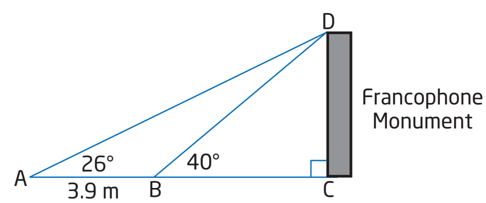
11. The Canadian Coast Guard Pacific Region is responsible for more than 27 000 km of coastline. The rotating spotlight from the Coast Guard ship can illuminate up to a distance of 250 m. An observer on the shore is 500 m from the ship. His line of sight to the ship makes an angle of 20° with the shoreline. What length of shoreline is illuminated by the spotlight?



12. A chandelier is suspended from a horizontal beam by two support chains. One of the chains is 3.6 m long and forms an angle of 62° with the beam. The second chain is 4.8 m long. What angle does the second chain make with the beam?



13. Nicolina wants to approximate the height of the Francophone Monument in Edmonton. From the low wall surrounding the statue, she measures the angle of elevation to the top of the monument to be 40°. She measures a distance 3.9 m farther away from the monument and measures the angle of elevation to be 26°. Determine the height of the Francophone Monument.



Did You Know?

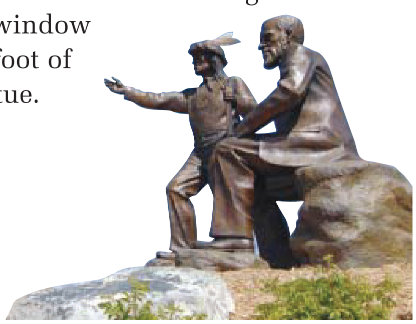
The Francophone Monument located at the Legislature Grounds in Edmonton represents the union of the fleur de lis and the wild rose. This monument celebrates the contribution of francophones to Alberta's heritage.

14. From the window of his hotel in Saskatoon, Max can see statues of Chief Whitecap of the Whitecap First Nation and John Lake, leader of the Temperance Colonists, who founded Saskatoon. The angle formed by Max's lines of sight to the top and to the foot of the statue of Chief Whitecap is 3°. The angle of depression of Max's line of sight to the top of the statue is 21°. The horizontal distance between Max and the front of the statue is 66 m.

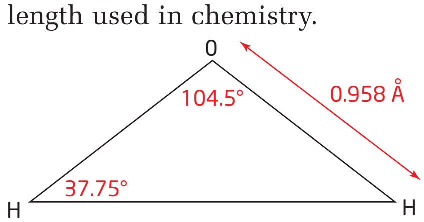
a) Sketch a diagram to represent this problem.

b) Determine the height of the statue of Chief Whitecap.

c) Determine the line-of-sight distance from where Max is standing at the window to the foot of the statue.

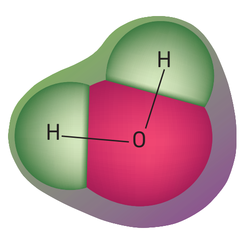


15. The chemical formula for water, H_2O, tells you that one molecule of water is made up of two atoms of hydrogen and one atom of oxygen bonded together. The nuclei of the atoms are separated by the distance shown, in angstroms. An angstrom is a unit of length used in chemistry.

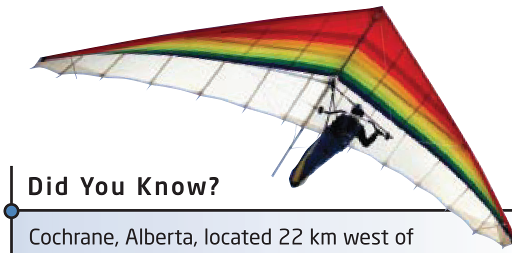


a) Determine the distance, in angstroms (Å), between the two hydrogen atoms.

b) Given that 1 Å = 0.01 mm, what is the distance between the two hydrogen atoms, in millimetres?



16. A hang-glider is a triangular parachute made of nylon or Dacron fabric. The pilot of a hang-glider flies through the air by finding updrafts and wind currents. The nose angle for a hang-glider may vary. The nose angle for the T2 high-performance glider ranges from 127° to 132°. If the length of the wing is 5.1 m, determine the greatest and least wingspans of the T2 glider.



Did You Know?

Cochrane, Alberta, located 22 km west of Calgary, is considered to be the perfect location for hang-gliding. The prevailing winds are heated as they cross the prairie. Then they are forced upward by hills, creating powerful thermals that produce the long flights so desired by flyers.

The Canadian record for the longest hang-gliding flight is a distance of 332.8 km, flown from east of Beiseker, Alberta, to northwest of Westlock, in May 1989, by Willi Muller.

17. On his trip to Somerset Island, Nunavut, Armand joined an informative tour near Fort Ross. During the group's first stop, Armand spotted a cairn at the top of a hill, a distance of 500 m away. The group made a second stop at the bottom of the hill. From this point, the cairn is 360 m away. The angle between the cairn, Armand's first stop, and his second stop is 35°.

a) Explain why there are two possible locations for Armand's second stop.

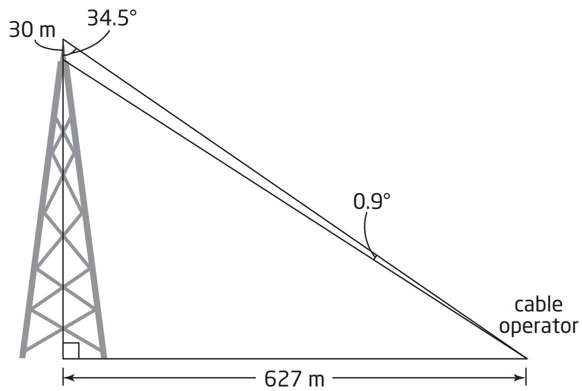
b) Sketch a diagram to illustrate each possible location.

c) Determine the possible distances between Armand's first and second stops.

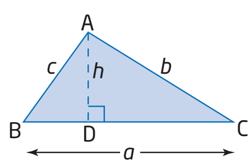
Did You Know?

A cairn is a pile of stones placed as a memorial, tomb, or landmark.

18. Radio towers, designed to support broadcasting and telecommunications, are among the tallest non-natural structures. Construction and maintenance of radio towers is rated as one of the most dangerous jobs in the world. To change the antenna of one of these towers, a crew sets up a system of pulleys. The diagram models the machinery and cable set-up. Suppose the height of the antenna is 30 m. Determine the total height of the structure.



19. Given an obtuse △ABC, copy and complete the table. Indicate the reasons for each step for the proof of the sine law.

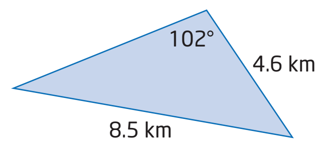


Statements	Reasons
$\sin C = \frac{h}{b}$ $\sin B = \frac{h}{c}$	
$h = b \sin C$ $h = c \sin B$	
$b \sin C = c \sin B$	
$\frac{\sin C}{c} = \frac{\sin B}{b}$	

Extend

20. Use the sine law to prove that if the measures of two angles of a triangle are equal, then the lengths of the sides opposite those angles are equal. Use a sketch in your explanation.

21. There are about 12 reported oil spills of 4000 L or more each day in Canada. Oil spills, such as occurred after the train derailment near Wabamun Lake, Alberta, can cause long-term ecological damage. To contain the spilled oil, floating booms are placed in the water. Suppose for the cleanup of the 734 000 L of oil at Wabamun, the floating booms used approximated an oblique triangle with the measurements shown. Determine the area of the oil spill at Wabamun.



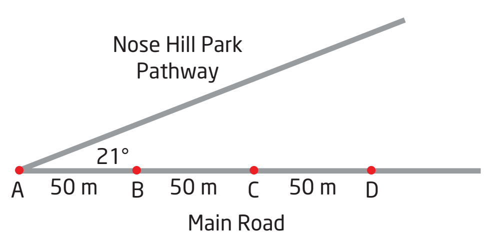
22. For each of the following, include a diagram in your solution.

a) Determine the range of values side a can have so that △ABC has two solutions if ∠A = 40° and b = 50.0 cm.

b) Determine the range of values side a can have so that △ABC has no solutions if ∠A = 56° and b = 125.7 cm.

c) △ABC has exactly one solution. If ∠A = 57° and b = 73.7 cm, what are the values of side a for which this is possible?

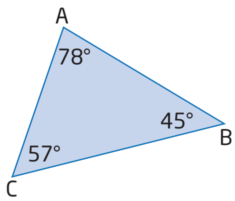
23. Shawna takes a pathway through Nose Hill Park in her Calgary neighbourhood. Street lights are placed 50 m apart on the main road, as shown. The light from each streetlight illuminates a distance along the ground of up to 60 m. Determine the distance from A to the farthest point on the pathway that is lit.



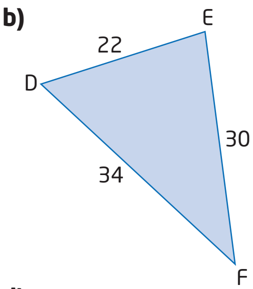
Create Connections

24. Explain why the sine law cannot be used to solve each triangle.

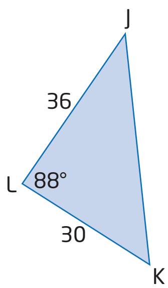
a)



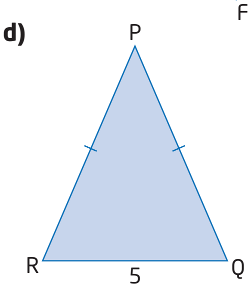
b)



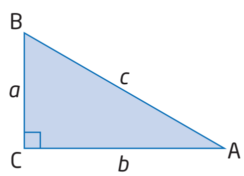
c)



d)



25. Explain how you could use a right △ABC to partially develop the sine law.



26. A golden triangle is an isosceles triangle in which the ratio of the length of the longer side to the length of the shorter side is the golden ratio, $\frac{\sqrt{5}+1}{2}:1$. The golden ratio is found in art, in math, and in architecture. In golden triangles, the vertex angle measures 36° and the two base angles measure 72°.

a) The triangle in Figure 1 is a golden triangle. The base measures 8 cm. Use the sine law to determine the length of the two equal sides of the triangle.

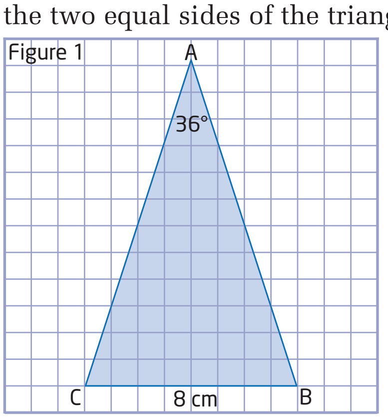


Figure 1

b) Use the golden ratio to determine the exact lengths of the two equal sides.

c) If you bisect one base angle of a golden triangle, you create another golden triangle similar to the first, as in Figure 2. Determine the length of side CD.

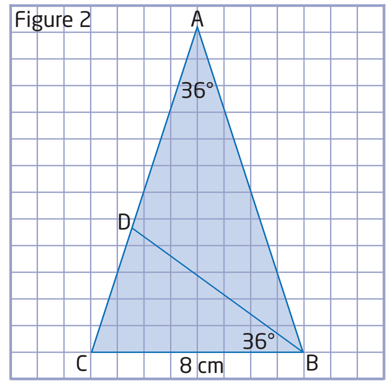


Figure 2

d) This pattern may be repeated, as in Figure 3. Determine the length of DE.

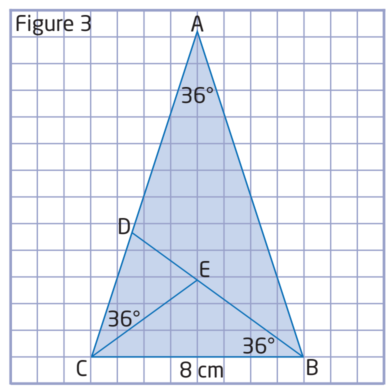


Figure 3

e) Describe how the spiral in Figure 4 is created.

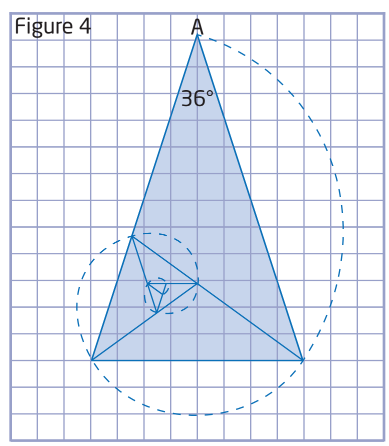
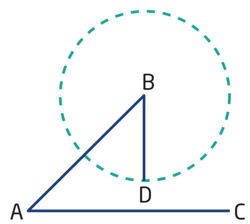


Figure 4

27. Complete a concept map to show the conditions necessary to be able to use the sine law to solve triangles.

28. MINI LAB Work with a partner to explore conditions for the ambiguous case of the sine law.



Materials

- ruler
- compass
- string
- scissors

Step 1 Draw a line segment AC. Draw a second line segment AB that forms an acute angle at A. Draw a circle, using point B as the centre, such that the circle does not touch the line segment AC. Draw a radius and label the point intersecting the circle D.

Step 2 Cut a piece of string that is the length of the radius BD. Hold one end at the centre of the circle and turn the other end through the arc of the circle.

a) Can a triangle be formed under these conditions?

b) Make a conjecture about the number of triangles formed and the conditions necessary for this situation.

Step 3 Extend the circle so that it just touches the line segment AC at point D. Cut a piece of string that is the length of the radius. Hold one end at the centre of the circle and turn the other end through the arc of the circle.

a) Can a triangle be formed under these conditions?

b) Make a conjecture about the number of triangles formed and the conditions necessary for this situation.

Step 4 Extend the circle so that it intersects line segment AC at two distinct points. Cut a piece of string that is the length of the radius. Hold one end at the centre of the circle and turn the other end through the arc of the circle.

a) Can a triangle be formed under these conditions?

b) Make a conjecture about the number of triangles formed and the conditions necessary for this situation.

Step 5 Cut a piece of string that is longer than the segment AB. Hold one end at B and turn the other end through the arc of a circle.

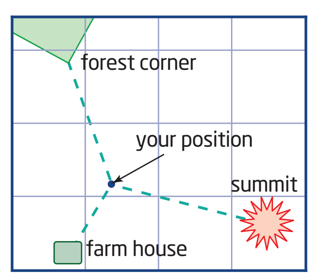
a) Can a triangle be formed under these conditions?

b) Make a conjecture about the number of triangles formed and the conditions necessary for this situation.

Step 6 Explain how varying the measure of ∠A would affect your conjectures.

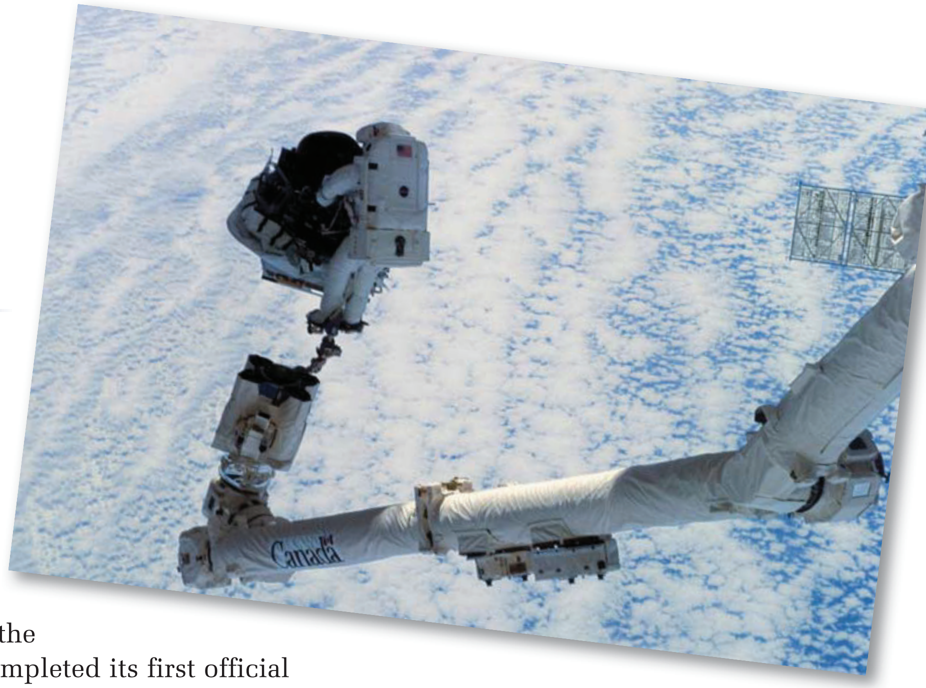
Project Corner Triangulation

- Triangulation is a method of determining your exact location based on your position relative to three other established locations using angle measures or bearings.
- Bearings are angles measured in degrees from the north line in a clockwise direction. A bearing usually has three digits. For instance, north is 000°, east is 090°, south is 180°, and southwest is 225°.
- A bearing of 045° is the same as N45°E.
- How could you use triangulation to help you determine the location of your resource?



2.4

The Cosine Law



Focus on…

- sketching a diagram and solving a problem using the cosine law
- recognizing when to use the cosine law to solve a given problem
- explaining the steps in the given proof of the cosine law

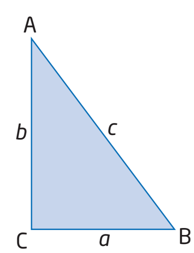
The Canadarm2, one of the three components of the Mobile Servicing System, is a major part of the Canadian space robotic system. It completed its first official construction job on the International Space Station in July 2001. The robotic arm can move equipment and assist astronauts working in space. The robotic manipulator is operated by controlling the angles of its joints. The final position of the arm can be calculated by using the trigonometric ratios of those angles.

Investigate the Cosine Law

Materials

- ruler
- protractor

1. a) Draw $\triangle ABC$, where $a = 3$ cm, $b = 4$ cm, and $c = 5$ cm.

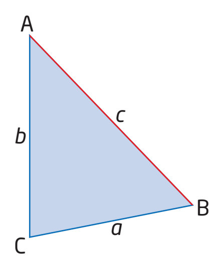


b) Determine the values of a^2, b^2, and c^2.

c) Compare the values of $a^2 + b^2$ and c^2. Which of the following is true?
- $a^2 + b^2 = c^2$
- $a^2 + b^2 > c^2$
- $a^2 + b^2 < c^2$

d) What is the measure of $\angle C$?

2. a) Draw an acute $\triangle ABC$.



b) Measure the lengths of sides a, b, and c.

c) Determine the values of a^2, b^2, and c^2.

d) Compare the values of $a^2 + b^2$ and c^2. Which of the following is true?
- $a^2 + b^2 > c^2$
- $a^2 + b^2 < c^2$

The cosine law relates the lengths of the sides of a given triangle to the cosine of one of its angles.

3. a) For $\triangle ABC$ given in step 1, determine the value of $2ab \cos C$.

b) Determine the value of $2ab \cos C$ for $\triangle ABC$ from step 2.

c) Copy and complete a table like the one started below. Record your results and collect data for the triangle drawn in step 2 from at least three other people.

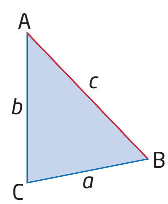
Triangle Side Lengths (cm)	c^2	$a^2 + b^2$	$2ab \cos C$
$a = 3$, $b = 4$, $c = 5$			
$a = ■$, $b = ■$, $c = ■$			

4. Consider the inequality you found to be true in step 2, for the relationship between the values of c^2 and $a^2 + b^2$. Explain how your results from step 3 might be used to turn the inequality into an equation. This relationship is known as the cosine law.

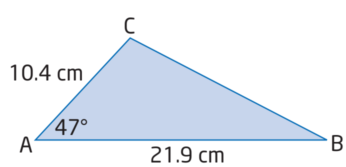
5. Draw $\triangle ABC$ in which $\angle C$ is obtuse. Measure its side lengths. Determine whether or not your equation from step 4 holds.

Reflect and Respond

6. The cosine law relates the lengths of the sides of a given triangle to the cosine of one of its angles. Under what conditions would you use the cosine law to solve a triangle?



7. Consider the triangle shown.



a) Is it possible to determine the length of side a using the sine law? Explain why or why not.

b) Describe how you could solve for side a.

8. How are the cosine law and the Pythagorean Theorem related?

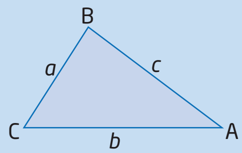
Link the Ideas

The Cosine Law

cosine law

- if a, b, c are the sides of a triangle and C is the angle opposite c, the cosine law is $c^2 = a^2 + b^2 - 2ab \cos C$

The **cosine law** describes the relationship between the cosine of an angle and the lengths of the three sides of any triangle.



For any $\triangle ABC$, where a, b, and c are the lengths of the sides opposite to $\angle A$, $\angle B$, and $\angle C$, respectively, the cosine law states that

$c^2 = a^2 + b^2 - 2ab \cos C$

You can express the formula in different forms to find the lengths of the other sides of the triangle.

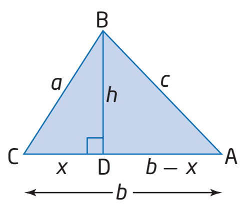
$a^2 = b^2 + c^2 - 2bc \cos A$

$b^2 = a^2 + c^2 - 2ac \cos B$

What patterns do you notice?

Proof

In $\triangle ABC$, draw an altitude h.



In $\triangle BCD$:

$\cos C = \frac{x}{a}$ $\quad\quad a^2 = h^2 + x^2$

$x = a \cos C$

In $\triangle ABD$, using the Pythagorean Theorem:

$c^2 = h^2 + (b - x)^2$

$c^2 = h^2 + b^2 - 2bx + x^2$ — Expand the binomial.

$c^2 = h^2 + x^2 + b^2 - 2bx$ — Why are the terms rearranged?

$c^2 = a^2 + b^2 - 2b(a \cos C)$ — Explain the substitutions.

$c^2 = a^2 + b^2 - 2ab \cos C$

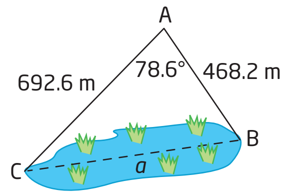
Example 1

Determine a Distance

A surveyor needs to find the length of a swampy area near Fishing Lake, Manitoba. The surveyor sets up her transit at a point A. She measures the distance to one end of the swamp as 468.2 m, the distance to the opposite end of the swamp as 692.6 m, and the angle of sight between the two as 78.6°. Determine the length of the swampy area, to the nearest tenth of a metre.

Solution

Sketch a diagram to illustrate the problem.



Can you use the sine law or the Pythagorean Theorem to solve for a? Why or why not?

Use the cosine law.

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$a^2 = 692.6^2 + 468.2^2 - 2(692.6)(468.2)\cos 78.6°$$
$$a^2 = 570\ 715.205\ldots$$
$$a = \sqrt{570\ 715.205\ldots}$$
$$a = 755.456\ldots$$

The length of the swampy area is 755.5 m, to the nearest tenth of a metre.

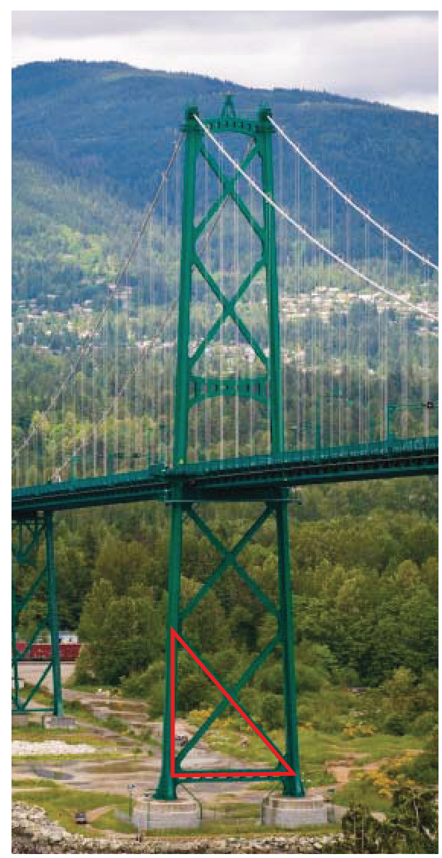
Your Turn

Nina wants to find the distance between two points, A and B, on opposite sides of a pond. She locates a point C that is 35.5 m from A and 48.8 m from B. If the angle at C is 54°, determine the distance AB, to the nearest tenth of a metre.

Example 2

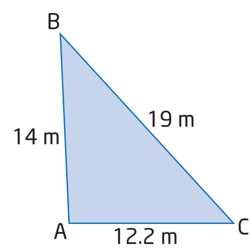
Determine an Angle

The Lions' Gate Bridge has been a Vancouver landmark since it opened in 1938. It is the longest suspension bridge in Western Canada. The bridge is strengthened by triangular braces. Suppose one brace has side lengths 14 m, 19 m, and 12.2 m. Determine the measure of the angle opposite the 14-m side, to the nearest degree.



Solution

Sketch a diagram to illustrate the situation.



Use the cosine law: $c^2 = a^2 + b^2 - 2ab \cos C$

Method 1: Substitute Directly

$$c^2 = a^2 + b^2 - 2ab \cos C$$
$$14^2 = 19^2 + 12.2^2 - 2(19)(12.2) \cos C$$
$$196 = 361 + 148.84 - 463.6 \cos C$$
$$196 = 509.84 - 463.6 \cos C$$
$$196 - 509.84 = -463.6 \cos C$$
$$-313.84 = -463.6 \cos C$$
$$\frac{-313.84}{-463.6} = \cos C$$
$$\cos^{-1}\left(\frac{313.84}{463.6}\right) = \angle C$$
$$47.393\ldots = \angle C$$

The measure of the angle opposite the 14-m side is approximately 47°.

Method 2: Rearrange the Formula to Solve for cos C

$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$2ab\cos C = a^2 + b^2 - c^2$$
$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$
$$\cos C = \frac{19^2 + 12.2^2 - 14^2}{2(19)(12.2)}$$
$$\angle C = \cos^{-1}\left(\frac{19^2 + 12.2^2 - 14^2}{2(19)(12.2)}\right)$$
$$\angle C = 47.393\ldots$$

The measure of the angle opposite the 14-m side is approximately 47°.

How can you write a similar formula for ∠A or ∠B?

Your Turn

A triangular brace has side lengths 14 m, 18 m, and 22 m. Determine the measure of the angle opposite the 18-m side, to the nearest degree.

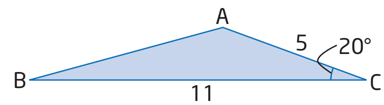
Example 3

Solve a Triangle

In △ABC, $a = 11$, $b = 5$, and ∠C = 20°. Sketch a diagram and determine the length of the unknown side and the measures of the unknown angles, to the nearest tenth.

Solution

Sketch a diagram of the triangle.



List the measures.

∠A = ■	$a = 11$
∠B = ■	$b = 5$
∠C = 20°	c = ■

Use the cosine law to solve for c.

Could you use the sine law? Explain.

$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$c^2 = 11^2 + 5^2 - 2(11)(5)\cos 20°$$
$$c^2 = 42.633\ldots$$
$$c = 6.529\ldots$$

To solve for the angles, you could use either the cosine law or the sine law.

For ∠A:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$
$$\cos A = \frac{5^2 + (6.529\ldots)^2 - 11^2}{2(5)(6.529\ldots)}$$
$$\angle A = \cos^{-1}\left(\frac{5^2 + (6.529\ldots)^2 - 11^2}{2(5)(6.529\ldots)}\right)$$
$$\angle A = 144.816\ldots$$

For c, use your calculator value.

The measure of ∠A is approximately 144.8°.

Use the angle sum of a triangle to determine ∠C.

$$\angle C = 180° - (20° + 144.8°)$$
$$\angle C = 15.2°$$

Could you find the measure of ∠A or ∠C using the sine law? If so, which is better to find first?

Did You Know?

As a general rule, it is better to use the cosine law to find angles, since the inverse cosine function ($\cos^{-1}$) on a calculator will display an obtuse angle when the cosine ratio is negative.

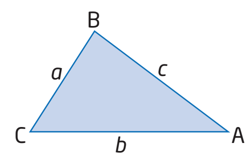
The six parts of the triangle are as follows:
∠A = 144.8° $a = 11$
∠B = 15.2° $b = 5$
∠C = 20° $c = 6.5$

Your Turn

In △ABC, $a = 9$, $b = 7$, and ∠C = 33.6°. Sketch a diagram and determine the length of the unknown side and the measures of the unknown angles, to the nearest tenth.

Key Ideas

- Use the cosine law to find the length of an unknown side of any triangle when you know the lengths of two sides and the measure of the angle between them.
- The cosine law states that for any △ABC, the following relationships exist:
  $a^2 = b^2 + c^2 - 2bc \cos A$
  $b^2 = a^2 + c^2 - 2ac \cos B$
  $c^2 = a^2 + b^2 - 2ab \cos C$



- Use the cosine law to find the measure of an unknown angle of any triangle when the lengths of three sides are known.
- Rearrange the cosine law to solve for a particular angle.
  For example, $\cos A = \frac{a^2 - b^2 - c^2}{-2bc}$ or $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$.
- Use the cosine law in conjunction with the sine law to solve a triangle.

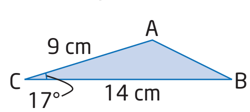
Check Your Understanding

Practise

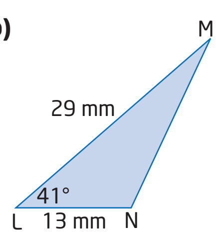
Where necessary, round lengths to the nearest tenth of a unit and angles to the nearest degree, unless otherwise stated.

1. Determine the length of the third side of each triangle.

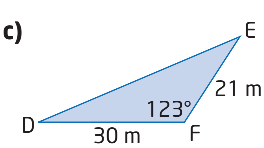
a)



b)

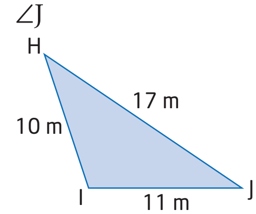


c)

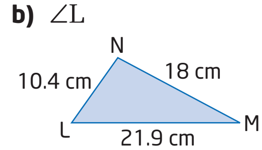


2. Determine the measure of the indicated angle.

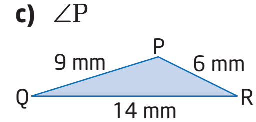
a) ∠J



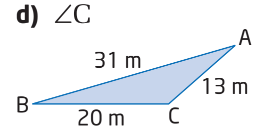
b) ∠L



c) ∠P

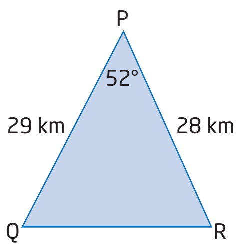


d) ∠C

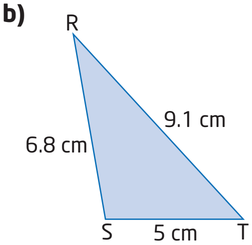


3. Determine the lengths of the unknown sides and the measures of the unknown angles.

a)



b)



4. Make a sketch to show the given information for each △ABC. Then, determine the indicated value.

a) AB = 24 cm, AC = 34 cm, and ∠A = 67°. Determine the length of BC.

b) AB = 15 m, BC = 8 m, and ∠B = 24°. Determine the length of AC.

c) AC = 10 cm, BC = 9 cm, and ∠C = 48°. Determine the length of AB.

d) AB = 9 m, AC = 12 m, and BC = 15 m. Determine the measure of ∠B.

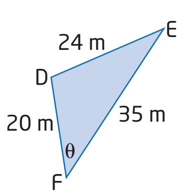
e) AB = 18.4 m, BC = 9.6 m, and AC = 10.8 m. Determine the measure of ∠A.

f) AB = 4.6 m, BC = 3.2 m, and AC = 2.5 m. Determine the measure of ∠C.

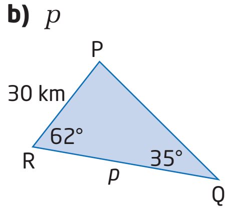
Apply

5. Would you use the sine law or the cosine law to determine each indicated side length or angle measure? Give reasons for your choice.

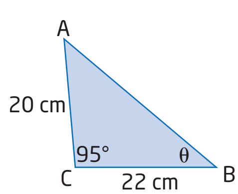
a) ∠F



b) p

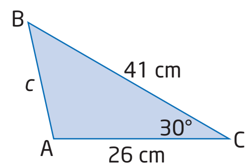


c) ∠B

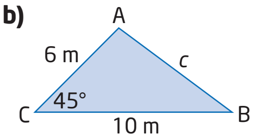


6. Determine the length of side c in each △ABC, to the nearest tenth.

a)



b)



7. In a parallelogram, the measure of the obtuse angle is 116°. The adjacent sides, containing the angle, measure 40 cm and 22 cm, respectively. Determine the length of the longest diagonal.

8. The longest tunnel in North America could be built through the mountains of the Kicking Horse Canyon, near Golden, British Columbia. The tunnel would be on the Trans-Canada highway connecting the Prairies with the west coast. Suppose the surveying team selected a point A, 3000 m away from the proposed tunnel entrance and 2000 m from the tunnel exit. If ∠A is measured as 67.7°, determine the length of the tunnel, to the nearest metre.

Web Link

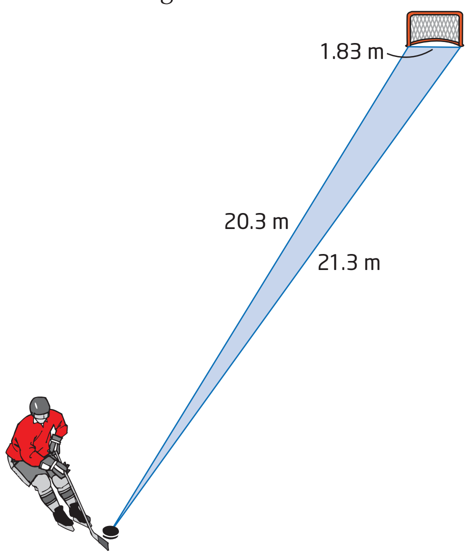
To learn more about the history of the Kicking Horse Pass and proposed plans for a new tunnel, go to www.mhrprecalc11.ca and follow the links.

9. Thousands of Canadians are active in sailing clubs. In the Paralympic Games, there are competitions in the single-handed, double-handed, and three-person categories. A sailing race course often follows a triangular route around three buoys. Suppose the distances between the buoys of a triangular course are 8.56 km, 5.93 km, and 10.24 km. Determine the measure of the angle at each of the buoys.

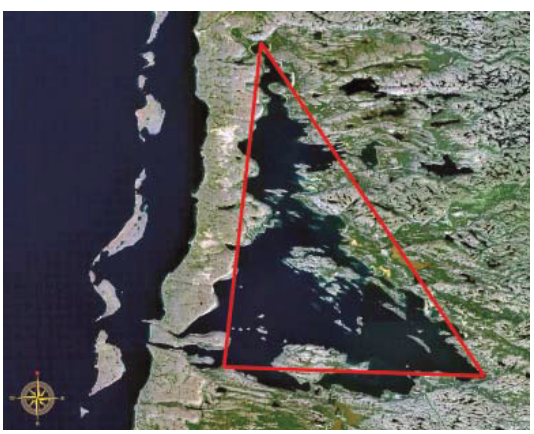
Did You Know?

Single-handed sailing means that one person sails the boat. Double-handed refers to two people. Buoys are floating markers, anchored to keep them in place. The oldest record of buoys being used to warn of rock hazards is in the thirteenth century on the Guadalquivir River near Seville in Spain.

10. The Canadian women's national ice hockey team has won numerous international competitions, including gold medals at the 2002, 2006, and 2010 Winter Olympics. A player on the blue line shoots a puck toward the 1.83-m-wide net from a point 20.3 m from one goal post and 21.3 m from the other. Within what angle must she shoot to hit the net? Answer to the nearest tenth of a degree.

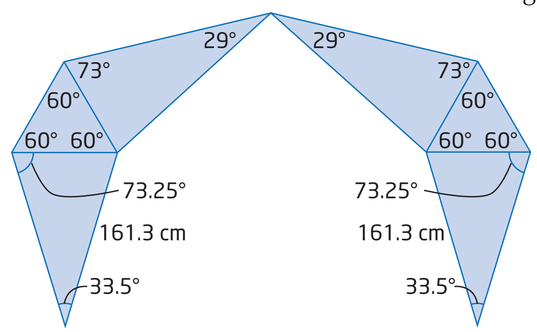


11. One of the best populated sea-run brook trout areas in Canada is Lac Guillaume-Delisle in northern Québec. Also known as Richmond Gulf, it is a large triangular-shaped lake. Suppose the sides forming the northern tip of the lake are 65 km and 85 km in length, and the angle at the northern tip is 7.8°. Determine the width of the lake at its base.



12. An aircraft-tracking station determines the distances from a helicopter to two aircraft as 50 km and 72 km. The angle between these two distances is 49°. Determine the distance between the two aircraft.

13. Tony Smith was born in 1912 in South Orange, New Jersey. As a child, Tony suffered from tuberculosis. He spent his time playing and creating with medicine boxes. His sculpture, Moondog, consists of several equilateral and isosceles triangles that combine art and math. Use the information provided on the diagram to determine the maximum width of Moondog.



Moondog by Tony Smith

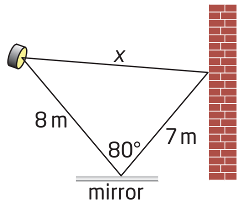
14. Julia and Isaac are backpacking in Banff National Park. They walk 8 km from their base camp heading N42°E. After lunch, they change direction to a bearing of 137° and walk another 5 km.

a) Sketch a diagram of their route.

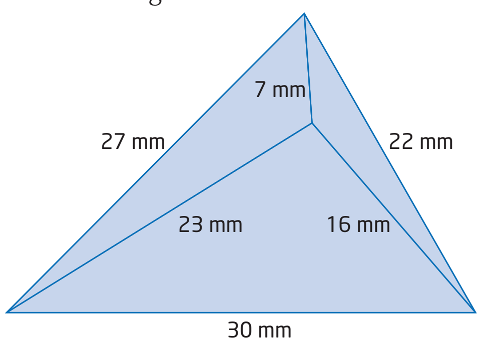
b) How far are Julia and Isaac from their base camp?

c) At what bearing must they travel to return to their base camp?

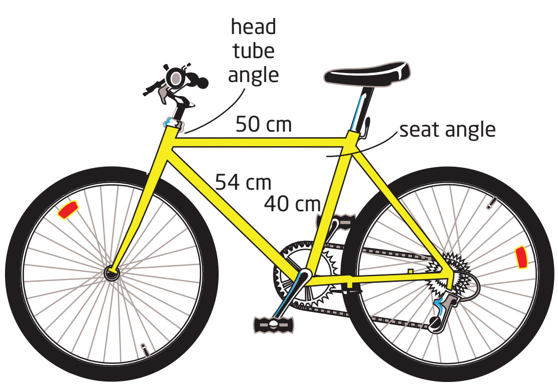
15. A spotlight is 8 m away from a mirror on the floor. A beam of light shines into the mirror, forming an angle of 80° with the reflected light. The light is reflected a distance of 7 m to shine onto a wall. Determine the distance from the spotlight to the point where the light is reflected onto the wall.



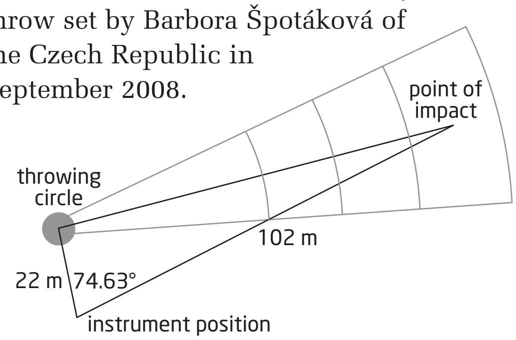
16. Erica created this design for part of a company logo. She needs to determine the accuracy of the side lengths. Explain how you could use the cosine law to verify that the side lengths shown are correct.



17. The sport of mountain biking became popular in the 1970s. The mountain bike was designed for off-road cycling. The geometry of the mountain bike contains two triangles designed for the safety of the rider. The seat angle and the head tube angle are critical angles that affect the position of the rider and the performance of the bike. Calculate the interior angles of the frame of the mountain bike shown.

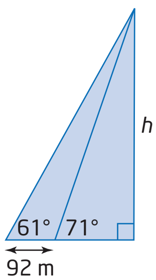


18. Distances in the throwing events at the Olympic games, traditionally measured with a tape measure, are now found with a piece of equipment called the Total Station. This instrument measures the angles and distances for events such as the shot put, the discus, and the javelin. Use the measurements shown to determine the distance, to the nearest hundredth of a metre, of the world record for the javelin throw set by Barbora Špotáková of the Czech Republic in September 2008.

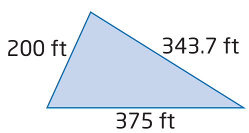


19. The Bermuda Triangle is an unmarked area in the Atlantic Ocean where there have been reports of unexplained disappearances of boats and planes and problems with radio communications. The triangle is an isosceles triangle with vertices at Miami, Florida, San Juan, Puerto Rico, and at the island of Bermuda. Miami is approximately 1660 km from both Bermuda and San Juan. If the angle formed by the sides of the triangle connecting Miami to Bermuda and Miami to San Juan is 55.5°, determine the distance from Bermuda to San Juan. Express your answer to the nearest kilometre.

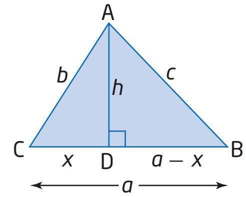
20. Della Falls in Strathcona Provincial Park on Vancouver Island is the highest waterfall in Canada. A surveyor measures the angle to the top of the falls to be 61°. He then moves in a direct line toward the falls a distance of 92 m. From this closer point, the angle to the top of the falls is 71°. Determine the height of Della Falls.



21. The floor of the Winnipeg Art Gallery is in the shape of a triangle with sides measuring 343.7 ft, 375 ft, and 200 ft. Determine the measures of the interior angles of the building.



22. Justify each step of the proof of the cosine law.



Statement	Reason
$c^2 = (a - x)^2 + h^2$	
$c^2 = a^2 - 2ax + x^2 + h^2$	
$b^2 = x^2 + h^2$	
$c^2 = a^2 - 2ax + b^2$	
$\cos C = \frac{x}{b}$	
$x = b \cos C$	
$c^2 = a^2 - 2a(b \cos C) + b^2$	
$c^2 = a^2 + b^2 - 2ab \cos C$	

Extend

23. Two ships leave port at 4 p.m. One is headed N38°E and is travelling at 11.5 km/h. The other is travelling at 13 km/h, with heading S47°E. How far apart are the two ships at 6 p.m.?

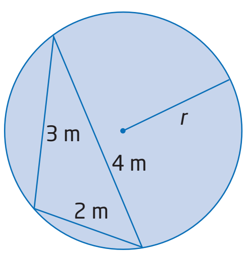
24. Is it possible to draw a triangle with side lengths 7 cm, 8 cm, and 16 cm? Explain why or why not. What happens when you use the cosine law with these numbers?

25. The hour and minute hands of a clock have lengths of 7.5 cm and 15.2 cm, respectively. Determine the straight-line distance between the tips of the hands at 1:30 p.m., if the hour hand travels 0.5° per minute and the minute hand travels 6° per minute.

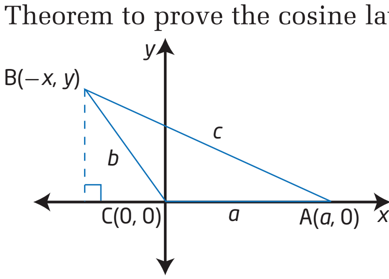
26. Graph A(−5, −4), B(8, 2), and C(2, 7) on a coordinate grid. Extend BC to intersect the y-axis at D. Find the measure of the interior angle ∠ABC and the measure of the exterior angle ∠ACD.

27. Researchers at Queen's University use a combination of genetics, bear tracks, and feces to estimate the numbers of polar bears in an area and gather information about their health, gender, size, and age. Researchers plan to set up hair traps around King William Island, Nunavut. The hair traps, which look like fences, will collect polar bear hair samples for analysis. Suppose the hair traps are set up in the form of △ABC, where ∠B = 40°, c = 40.4 km, and a = 45.9 km. Determine the area of the region.

28. If the sides of a triangle have lengths 2 m, 3 m, and 4 m, what is the radius of the circle circumscribing the triangle?

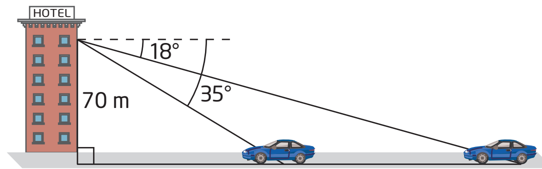


29. △ABC is placed on a Cartesian grid with ∠BCA in standard position. Point B has coordinates ($-x$, y). Use primary trigonometric ratios and the Pythagorean Theorem to prove the cosine law.



Create Connections

30. The Delta Regina Hotel is the tallest building in Saskatchewan. From the window of her room on the top floor, at a height of 70 m above the ground, Rian observes a car moving toward the hotel. If the angle of depression of the car changes from 18° to 35° during the time Rian is observing it, determine the distance the car has travelled.



31. Given △ABC where ∠C = 90°, a = 12.2 cm, b = 8.9 cm, complete the following.

a) Use the cosine law to find c^2.

b) Use the Pythagorean Theorem to find c^2.

c) Compare and contrast the cosine law with the Pythagorean Theorem.

d) Explain why the two formulas are the same in a right triangle.

32. When solving triangles, the first step is choosing which method is best to begin with. Copy and complete the following table. Place the letter of the method beside the information given. There may be more than one answer.

A primary trigonometric ratios

B sine law

C cosine law

D none of the above

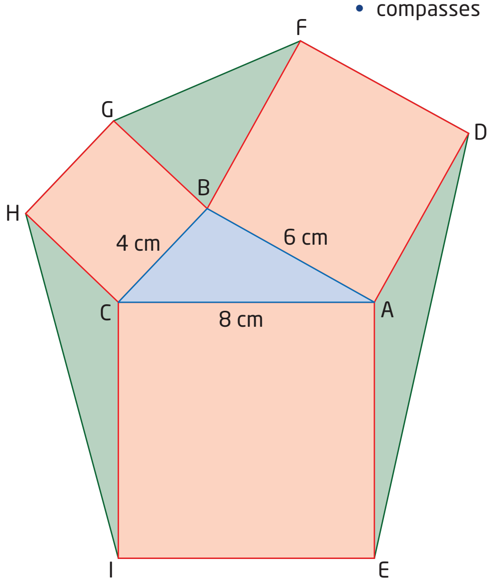
Concept Summary for Solving a Triangle

Given	Begin by Using the Method of
Right triangle	
Two angles and any side	
Three sides	
Three angles	
Two sides and the included angle	
Two sides and the angle opposite one of them	

33. MINI LAB

Materials

- ruler
- protractor
- compasses



Step 1 **a)** Construct △ABC with side lengths $a = 4$ cm, $b = 8$ cm, and $c = 6$ cm.

b) On each side of the triangle, construct a square.

c) Join the outside corners of the squares to form three new triangles.

Step 2 **a)** In △ABC, determine the measures of ∠A, ∠B, and ∠C.

b) Explain why pairs of angles, such as ∠ABC and ∠GBF, have a sum of 180°. Determine the measures of ∠GBF, ∠HCI, and ∠DAE.

c) Determine the lengths of the third sides, GF, DE, and HI, of △BGF, △ADE, and △CHI.

Step 3 For each of the four triangles, draw an altitude.

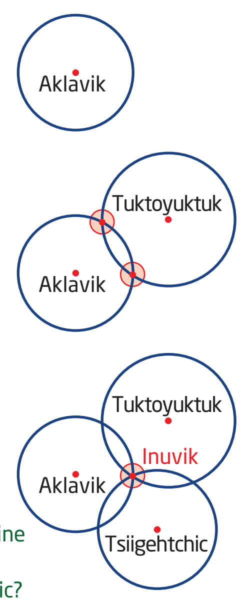
a) Use the sine ratio to determine the measure of each altitude.

b) Determine the area of each triangle.

Step 4 What do you notice about the areas of the triangles? Explain why you think this happens. Will the result be true for any △ABC?

Project Corner — Trilateration

- GPS receivers work on the principle of trilateration. The satellites circling Earth use 3-D trilateration to pinpoint locations. You can use 2-D trilateration to see how the principle works.
- Suppose you are 55 km from Aklavik, NT. Knowing this tells you that you are on a circle with radius 55 km centred at Aklavik.
- If you also know that you are 127 km from Tuktoyuktuk, you have two circles that intersect and you must be at one of these intersection points.
- If you are told that you are 132 km from Tsiigehtchic, a third circle will intersect with one of the other two points of intersection, telling you that your location is at Inuvik.
- How can you use the method of trilateration to pinpoint the location of your resource?



If you know the angle at Inuvik, how can you determine the distance between Tuktoyuktuk and Tsiigehtchic?

Chapter 2 Review

Where necessary, express lengths to the nearest tenth and angles to the nearest degree.

2.1 Angles in Standard Position, pages 74–87

1. Match each term with its definition from the choices below.

a) angle in standard position

b) reference angle

c) exact value

d) sine law

e) cosine law

f) terminal arm

g) ambiguous case

A a formula that relates the lengths of the sides of a triangle to the sine values of its angles

B a value that is not an approximation and may involve a radical

C the final position of the rotating arm of an angle in standard position

D the acute angle formed by the terminal arm and the *x*-axis

E an angle whose vertex is at the origin and whose arms are the *x*-axis and the terminal arm

F a formula that relates the lengths of the sides of a triangle to the cosine value of one of its angles

G a situation that is open to two or more interpretations

2. Sketch each angle in standard position. State which quadrant the angle terminates in and the measure of the reference angle.

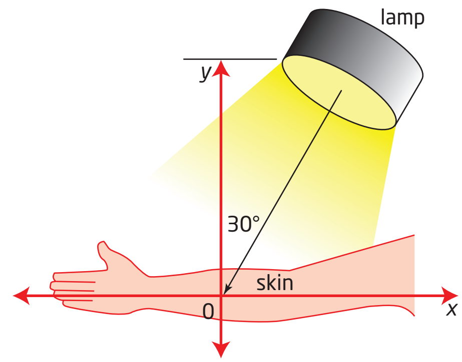
a) 200°

b) 130°

c) 20°

d) 330°

3. A heat lamp is placed above a patient's arm to relieve muscle pain. According to the diagram, would you consider the reference angle of the lamp to be 30°? Explain your answer.



4. Explain how to determine the measure of all angles in standard position, $0° \leq \theta < 360°$, that have 35° for their reference angle.

5. Determine the exact values of the sine, cosine, and tangent ratios for each angle.

a) 225°

b) 120°

c) 330°

d) 135°

2.2 Trigonometric Ratios of Any Angle, pages 88–99

6. The point Q(−3, 6) is on the terminal arm of an angle, θ.

a) Draw this angle in standard position.

b) Determine the exact distance from the origin to point Q.

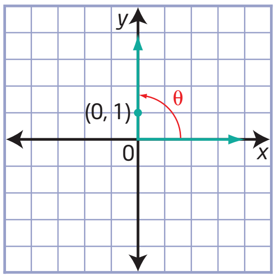
c) Determine the exact values for sin θ, cos θ, and tan θ.

d) Determine the value of θ.

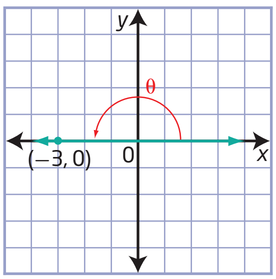
7. A reference angle has a terminal arm that passes through the point P(2, −5). Identify the coordinates of a corresponding point on the terminal arm of three angles in standard position that have the same reference angle.

8. Determine the values of the primary trigonometric ratios of θ in each case.

a)



b)



9. Determine the exact value of the other two primary trigonometric ratios given each of the following.

a) $\sin \theta = -\frac{3}{5}$, $\cos \theta < 0$

b) $\cos \theta = \frac{1}{3}$, $\tan \theta < 0$

c) $\tan \theta = \frac{12}{5}$, $\sin \theta > 0$

10. Solve for all values of θ, $0° \leq \theta < 360°$, given each trigonometric ratio value.

a) $\tan \theta = -1.1918$

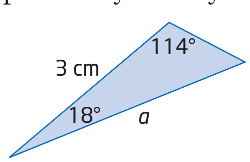
b) $\sin \theta = -0.3420$

c) $\cos \theta = 0.3420$

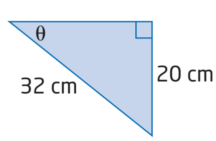
2.3 The Sine Law, pages 100–113

11. Does each triangle contain sufficient information for you to determine the unknown variable using the sine law? Explain why or why not.

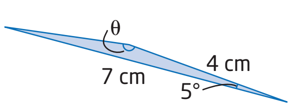
a)



b)

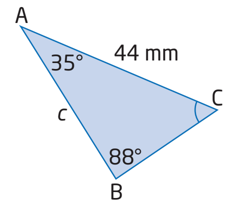


c)

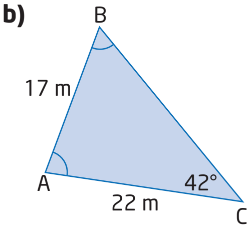


12. Determine the length(s) of the indicated side(s) and the measure(s) of the indicated angle(s) in each triangle.

a)



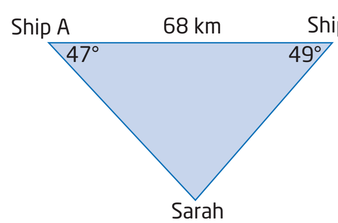
b)



13. In △PQR, ∠P = 63.5°, ∠Q = 51.2°, and *r* = 6.3 cm. Sketch a diagram and find the measures of the unknown sides and angle.

14. In travelling to Jasper from Edmonton, you notice a mountain directly in front of you. The angle of elevation to the peak is 4.1°. When you are 21 km closer to the mountain, the angle of elevation is 8.7°. Determine the approximate height of the mountain.

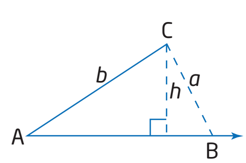
15. Sarah runs a deep-sea-fishing charter. On one of her expeditions, she has travelled 40 km from port when engine trouble occurs. There are two Search and Rescue (SAR) ships, as shown below.



a) Which ship is closer to Sarah? Use the sine law to determine her distance from that ship.

b) Verify your answer in part a) by using primary trigonometric ratios.

16. Given the measure of ∠A and the length of side *b* in △ABC, explain the restrictions on the lengths of sides *a* and *b* for the problem to have no solution, one solution, and two solutions.



17. A passenger jet is at cruising altitude heading east at 720 km/h. The pilot, wishing to avoid a thunderstorm, changes course by heading N70°E. The plane travels in this direction for 1 h, before turning to head toward the original path. After 30 min, the jet makes another turn onto its original path.

a) Sketch a diagram to represent the distances travelled by the jet to avoid the thunderstorm.

b) What heading, east of south, did the plane take, after avoiding the storm, to head back toward the original flight path?

c) At what distance, east of the point where it changed course, did the jet resume its original path?

2.4 The Cosine Law, pages 114–125

18. Explain why each set of information does not describe a triangle that can be solved.

a) $a = 7$, $b = 2$, $c = 4$

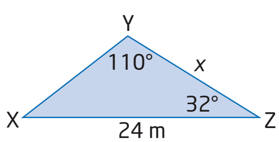
b) $\angle A = 85°$, $b = 10$, $\angle C = 98°$

c) $a = 12$, $b = 20$, $c = 8$

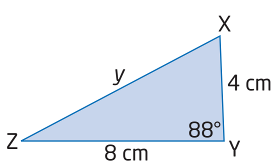
d) $\angle A = 65°$, $\angle B = 82°$, $\angle C = 35°$

19. Would you use the sine law or the cosine law to find each indicated side length? Explain your reasoning.

a)

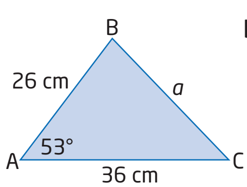


b)

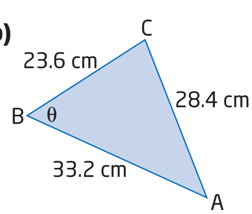


20. Determine the value of the indicated variable.

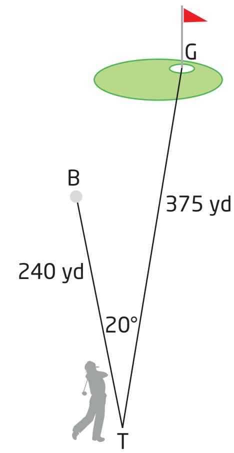
a)



b)



21. The 12th hole at a golf course is a 375-yd straightaway par 4. When Darla tees off, the ball travels 20° to the left of the line from the tee to the hole. The ball stops 240 yd from the tee (point B). Determine how far the ball is from the centre of the hole.



22. Sketch a diagram of each triangle and solve for the indicated value(s).

a) In △ABC, AB = 18.4 m, BC = 9.6 m, and AC = 10.8 m. Determine the measure of ∠A.

b) In △ABC, AC = 10 cm, BC = 9 cm, and ∠C = 48°. Determine the length of AB.

c) Solve △ABC, given that AB = 15 m, BC = 8 m, and ∠B = 24°.

23. Two boats leave a dock at the same time. Each travels in a straight line but in different directions. The angle between their courses measures 54°. One boat travels at 48 km/h and the other travels at 53.6 km/h.

a) Sketch a diagram to represent the situation.

b) How far apart are the two boats after 4 h?

24. The sides of a parallelogram are 4 cm and 6 cm in length. One angle of the parallelogram is 58° and a second angle is 122°.

a) Sketch a diagram to show the given information.

b) Determine the lengths of the two diagonals of the parallelogram.

Chapter 2 Practice Test

Multiple Choice

For #1 to #5, choose the best answer.

1. Which angle in standard position has a different reference angle than all the others?

A 125° **B** 155°

C 205° **D** 335°

2. Which angle in standard position does not have a reference angle of 55°?

A 35° **B** 125°

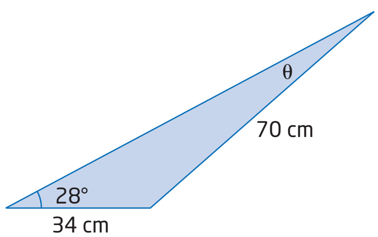
C 235° **D** 305°

3. Which is the exact value of cos 150°?

A $\frac{1}{2}$ **B** $\frac{\sqrt{3}}{2}$

C $-\frac{\sqrt{3}}{2}$ **D** $-\frac{1}{2}$

4. The equation that could be used to determine the measure of angle θ is



A $\frac{\sin \theta}{70} = \frac{\sin 28°}{34}$

B $\frac{\sin \theta}{34} = \frac{\sin 28°}{70}$

C $\cos \theta = \frac{70^2 + 34^2 - 28^2}{2(70)(34)}$

D $\theta^2 = 34^2 + 70^2 - 2(34)(70)\cos 28°$

5. For which of these triangles must you consider the ambiguous case?

A In △ABC, $a = 16$ cm, $b = 12$ cm, and $c = 5$ cm.

B In △DEF, ∠D = 112°, $e = 110$ km, and $f = 65$ km.

C In △ABC, ∠B = 35°, $a = 27$ m, and $b = 21$ m.

D In △DEF, ∠D = 108°, ∠E = 52°, and $f = 15$ cm.

Short Answer

6. The point P(2, b) is on the terminal arm of an angle, θ, in standard position. If $\cos \theta = \frac{1}{\sqrt{10}}$ and tan θ is negative, what is the value of b?

7. Oak Bay in Victoria, is in the direction of N57°E from Ross Bay. A sailboat leaves Ross Bay in the direction of N79°E. After sailing for 1.9 km, the sailboat turns and travels 1.1 km to reach Oak Bay.

a) Sketch a diagram to represent the situation.

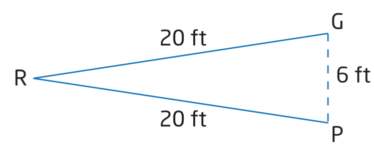
b) What is the distance between Ross Bay and Oak Bay?

8. In △ABC, $a = 10$, $b = 16$, and ∠A = 30°.

a) How many distinct triangles can be drawn given these measurements?

b) Determine the unknown measures in △ABC.

9. Rudy is 20 ft from each goal post when he shoots the puck along the ice toward the goal. The goal is 6 ft wide. Within what angle must he fire the puck to have a hope of scoring a goal?

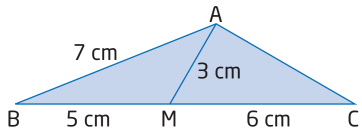


10. In △PQR, ∠P = 56°, $p = 10$ cm, and $q = 12$ cm.

a) Sketch a diagram of the triangle.

b) Determine the length of the unknown side and the measures of the unknown angles.

11. In $\triangle ABC$, M is a point on BC such that BM = 5 cm and MC = 6 cm. If AM = 3 cm and AB = 7 cm, determine the length of AC.

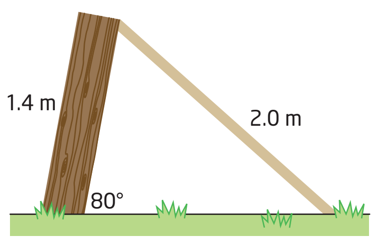


12. A fence that is 1.4 m tall has started to lean and now makes an angle of 80° with the ground. A 2.0-m board is jammed between the top of the fence and the ground to prop the fence up.

a) What angle, to the nearest degree, does the board make with the ground?

b) What angle, to the nearest degree, does the board make with the top of the fence?

c) How far, to the nearest tenth of a metre, is the bottom of the board from the base of the fence?



Extended Response

13. Explain, using examples, how to determine all angles $0° \leq \theta < 360°$ with a given reference angle θ_R.

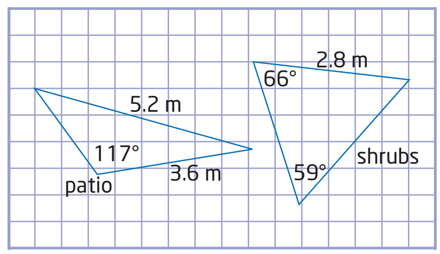
14. A softball diamond is a square measuring 70 ft on each side, with home plate and the three bases as vertices. The centre of the pitcher's mound is located 50 ft from home plate on a direct line from home to second base.

a) Sketch a diagram to represent the given information.

b) Show that first base and second base are not the same distance from the pitcher's mound.

15. Describe when to use the sine law and when to use the cosine law to solve a given problem.

16. As part of her landscaping course at the Northern Alberta Institute of Technology, Justine spends a summer redesigning her aunt's backyard. She chooses triangular shapes for her theme. Justine knows some of the measurements she needs in her design. Determine the unknown side lengths in each triangle.



17. The North Shore Rescue (NSR) is a mountain search and rescue team based in Vancouver. On one of their training exercises, the team splits up into two groups. From their starting point, the groups head off at an angle of 129° to each other. The Alpha group walks east for 3.8 km before losing radio contact with the Beta group. If their two-way radios have a range of 6.2 km, how far could the Beta group walk before losing radio contact with the Alpha group?