Chapter 6

Answers, Hints, Solutions

6.1 Limits

1. (a) 20

   (b) Does not exist

   (c) 0

   (d) 100

   (e) Does not exist. Consider the domain of $g(x) = \sqrt{-x^2 + 20x - 100} = \sqrt{-(x-10)^2}$.

2. $-8\ln 4$

3. 0. Note the exponential function in the denominator.

4. $\frac{3}{5}$. Divide the numerator and denominator by the highest power.

5. $\frac{5}{2}$

6. 3

7. 2

8. 0. What is the value of $3x + |1 - 3x|$ if $x < \frac{1}{3}$?

9. 1

10. $\frac{3}{\sqrt{2}}$

11. $-\frac{2}{3}$

12. ∞. Note that $x^2 - 1 = (x-1)(x+1)$.

13. -2. Which statement is true for $x < 1$: $|x-1| = x-1$ or $|x-1| = 1-x$?

14. (a) 1.5

    (b) -1.5

    (c) No. The left-hand limit and the right-hand limit are not equal.

15. Does not exist

16. $\frac{1}{8}$. Rationalize the numerator.

17. $\frac{1}{12}$. Note that $x - 8 = (\sqrt[3]{x} - 2)(\sqrt[3]{x^2} + 2\sqrt[3]{x} + 4)$.

18. $a = b = 4$. Rationalize the numerator. Choose the value of b so that x becomes

a factor in the numerator.

19. $\frac{1}{12}$. Note that $x-8=(\sqrt[3]{x}-2)(\sqrt[3]{x^2}+2\sqrt[3]{x}+4)$.

20. $\frac{1}{2}$. Rationalize the numerator.

21. $-\frac{3}{2}$. Rationalize the numerator. Note that $x \to -\infty$ and use the fact that if $x<0$ then $x=-\sqrt{x^2}$.

22. $-\frac{1}{2}$

23. $\frac{3}{2}$

24. Since the denominator approaches 0 as $x \to -2$, the necessary condition for this limit to exist is that the numerator approaches 0 as $x \to -2$. Thus we solve $4b-30+15+b=0$ to obtain $b=3$. $\lim_{x\to -2} \frac{3x^2+15x+18}{x^2+x-2} = -1$.

25. $a=4$. Write $f(x)=x+\frac{(a-1)x+5}{x+1}$.

26. $\lim_{x\to\infty} \frac{\ln x}{x} = 0$.

27. From $\lim_{x\to 4}(x+2)=6$ and $\lim_{x\to 4}(x^2-10)=6$, by the Squeeze Theorem, it follows that $\lim_{x\to 4} f(x) = 6$.

28. 1

29. From the fact that $|\sin(1/x)| \le 1$ for all $x \ne 0$ and the fact that the function $y=e^x$ is increasing conclude that $e^{-1} \le e^{\sin(1/x)} \le e$ for all $x \ne 0$. Thus $e^{-1}\cdot\sqrt{x} \le \sqrt{x}e^{\sin(1/x)} \le e\cdot\sqrt{x}$ for all $x>0$. By the Squeeze Theorem, $\lim_{x\to 0^+}\left(\sqrt{x}e^{\sin(1/x)}\right)=0$.

30. 0. Squeeze Theorem.

31. 0. Squeeze Theorem.

32. 0. Squeeze Theorem.

33. $-\infty$.

34. 0.

35. ∞.

36. $\frac{76}{45}$. This is the case "0/0". Apply L'Hospital's rule.

37. $\frac{1}{2}$. Write $\frac{1}{2}\cdot\left(\frac{\sin x}{x}\right)^{100}\cdot\frac{2x}{\sin 2x}$.

38. 7. Write $7\cdot\left(\frac{x}{\sin x}\right)^{101}\cdot\frac{\sin 7x}{7x}$.

39. 7.

40. $\frac{3}{5}$. This is the case "0/0". Apply L'Hospital's rule.

41. $\frac{3}{5}$.

42. 0. Write $x^2\cdot\frac{x}{\sin x}\cdot\sin\left(\frac{1}{x^2}\right)$.

43. Does not exist. $\frac{\sin x}{2|x|}\cdot\frac{1}{\sqrt{\frac{\sin 4x}{4x}}}$.

44. $\frac{1}{2}$. Write $\frac{1-\cos x}{x^2}\cdot\frac{x}{\sin x}$.

45. 1. Substitute $t=\frac{1}{x}$.

46. 0. This is the case "$\infty-\infty$". Write $\frac{x-\sin x}{x\sin x}$ and apply L'Hospital's rule.

47. $\frac{1}{6}$

48. 0. This is the case "$0\cdot\infty$". Write $\frac{\ln\sin x}{\frac{1}{\sin x}}$ and apply L'Hospital's rule.

49. 0. This is the case "∞/∞". Apply L'Hospital's rule.

50. 0.

51. 0.

52. 0.

53. 1. This is the case "0/0". Write $\dfrac{\ln(1+x)}{x}$ and apply L'Hospital's rule.

54. $\dfrac{3}{2}$. Use properties of logarithms first.

55. $\dfrac{3}{2}$.

56. $\ln 2$. The denominator approaches 2.

57. 0, This is the case "0/0". Apply L'Hospital's rule.

58. $-\dfrac{1}{\pi^2}$

59. ∞. This is the case "$\infty - \infty$". Write $\dfrac{\sin x - x^2 \cos x}{x^2 \sin x}$ and apply L'Hospital's rule.

60. $e^{\frac{1}{2}}$. This is the case "1^{∞}". Write $e^{\frac{\ln \cosh x}{x^2}}$. Apply L'Hospital's rule and use the fact that the exponential function $f(x) = e^x$ is continuous.

61. 1. This is the case "0^0". Write $x^x = e^{x \ln x} = e^{\frac{\ln x}{x^{-1}}}$. Apply L'Hospital's rule and use the fact that the exponential function $f(x) = e^x$ is continuous.

62. 1.

63. 1.

64. e.

65. 1. This is the case "∞^0".

66. 1.

67. e^3.

68. 0.

69. 1. Write $e^{x \ln \frac{x}{x+1}} = e^{x \ln x} \cdot e^{-x \ln(x+1)}$ and make your conclusion.

70. $e^{\frac{1}{e}}$.

71. 1.

72. 1. Squeeze Theorem.

73. e^{-2}. Write $\left((1-2x)^{-\frac{1}{2x}}\right)^{-2}$.

74. $e^{\frac{7}{5}}$. Write $\left((1+7x)^{\frac{1}{7x}}\right)^{\frac{7}{5}}$.

75. $e^{\frac{3}{8}}$. Write $\left((1+3x)^{\frac{1}{3x}}\right)^{\frac{3}{8}}$.

76. $e^{\frac{3}{2}}$. Write $\left(\left(1+\frac{x}{2}\right)^{\frac{2}{x}}\right)^{\frac{3}{2}}$.

77. 10. Use the fact that
$L = \lim_{n\to\infty} x_n$
to conclude $L^2 = 100$.
Can L be negative?

78. (a) $\dfrac{1}{2}$. Write $\dfrac{2\sin^2 \frac{x}{2}}{x^2}$, or use L'Hospital's rule.
 (b) 0.
 (c) Does not exist. Note that $f(x) = \arcsin x$ is defined on $[-1, 1]$.

79. (a) Does not exist. Note that $\lim_{h\to 0} \sqrt[4]{16+h} = 2$.
 (b) $-\dfrac{1}{\pi}$. Use L'Hospital's rule.
 (c) 1. Divide the numerator and denominator by u.

(d) e^{-2}.

(e) $\frac{1}{2}$

(f) ∞. Think, exponential vs. polynomial.

Consider $\lim_{x\to 0} \frac{xf(x)}{x|x|}$; (b) $\frac{1}{3}$;

(c) $-\frac{1}{4}$. Note that $x<0$; (d) e.

80. (a) $\frac{1}{2}$; (b) $\frac{7}{3}$; (c) 1; (d) 0; (e) $\frac{\sin 3 - 3}{27}$.

81. (a) Does not exist.

82. (a) $\frac{1}{4}$; (b) -1; (c) 0.

83. (a) $-\infty$; (b) 3; (c) 2; (d) 0.

84. Let $\varepsilon > 0$ be given. We need to find $\delta = \delta(\varepsilon) > 0$ such that $|x-0| < \delta \Rightarrow |x^3-0| < \varepsilon$, what is the same as $|x| < \delta \Rightarrow |x^3| < \varepsilon$. Clearly, we can take $\delta = \sqrt[3]{\varepsilon}$. Indeed. For any $\varepsilon > 0$ we have that $|x| < \sqrt[3]{\varepsilon} \Rightarrow |x|^3 = |x^3| < \varepsilon$ and, by definition, $\lim_{x\to 0} x^3 = 0$.

85. (c) For any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $|x-1| < \delta \Rightarrow |2x^2 - 2| < \varepsilon$.

86. $\lim_{h\to 0} \frac{f(x+h) - f(x-h)}{2h} = \lim_{h\to 0} \frac{f'(x+h) + f'(x-h)}{2}$ and, since f' is continuous, $\lim_{h\to 0} f'(x+h) = \lim_{h\to 0} f'(x-h) = f'(x)$.

6.2 Continuity

1. $c = \pi$. Solve $\lim_{x\to\pi^-} f(x) = \lim_{x\to\pi^-} f(x)$ for c. See Figure 6.1.

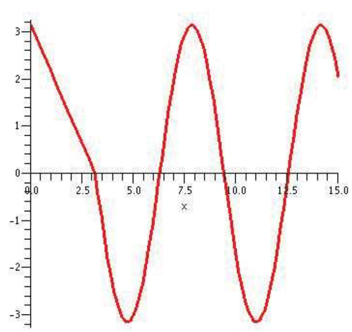

Figure 6.1: $c = \pi$

2. Let $f(x) = 2^x - \frac{10}{x}$. Note that the domain of f is the set $\mathbb{R}\backslash\{0\}$ and that on its domain, as a sum of two continuous function, f is continuous.

(a) Since f is continuous on $(0, \infty)$ and since $\lim_{x \to 0^+} f(x) = -\infty$ and $\lim_{x \to \infty} f(x) = \infty$ by the the Intermediate Value Property there is $a \in (0, \infty)$ such that $f(a) = 0$.

(b) For all $x \in (-\infty, 0)$ we have that $\frac{10}{x} < 0$ which implies that for all $x \in (-\infty, 0)$ we have that all $f(x) > 0$.

3. See Figure 6.2.

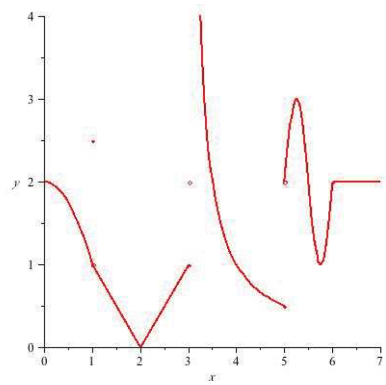

Figure 6.2: Piecewise Defined Function

(a) (i) False; (ii)True; (b) (i) Yes; (ii) Yes; (c) (i) No; (ii) No.

4. See Figure 6.3.

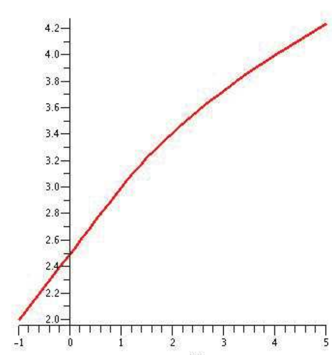

Figure 6.3: Continuous Function

(a) Check that $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$.

(b) $\frac{1}{2}$. Note $\lim_{x \to 1^-} \frac{\frac{5+x}{2} - 3}{x - 1} = \frac{1}{2}$ and $\lim_{x \to 1^+} \frac{(2 + \sqrt{x}) - 3}{x - 1} = \frac{1}{2}$.

5. $f(x) = \dfrac{x^2-9}{x-3}$ if $x \neq 3$ and $f(3) = 0$.

6.3 Miscellaneous

1. (a) $x = \dfrac{1-\ln\pi}{\ln\pi}$. (b) $x = -\dfrac{\log\log 2}{\log 3}$.

2. $(e,3) \cup (3,\infty)$.

3. (a) Give a definition of the limit. (b) Give a definition of a function continuous at a point. (c) A corner or a vertical tangent; $y = |x|$; $y = x^{\frac{1}{3}}$.

6.4 Derivatives

1. (b) $f'(3) = \lim\limits_{h\to 0} \dfrac{\frac{1}{2(3+h)-1} - \frac{1}{5}}{h} = \lim\limits_{h\to 0} \dfrac{-2}{5(5+2h)} = -0.08$. (c) $\lim\limits_{h\to 0} \dfrac{\sin^7\left(\frac{\pi}{6}+\frac{h}{2}\right) - \left(\frac{1}{2}\right)^7}{h} = \dfrac{d}{dx}(\sin^7 \dfrac{x}{2})\Big|_{x=\frac{\pi}{3}} = \dfrac{7}{2}\cdot\sin^6\dfrac{\pi}{6}\cdot\cos\dfrac{\pi}{6} = \dfrac{7\sqrt{3}}{256}$.

2. Let $|I(x)| \leq M$ for all $x \in \mathbb{R}$. Then for any $h \neq 0$, $\left|\dfrac{h^2 I(h)}{h}\right| = |hI(h)|$. Use the Squeeze Theorem to conclude that f is differentiable at $x = 0$.

3. $f'(2) = \lim\limits_{x\to 2} \dfrac{x + \frac{1}{x} - \frac{5}{2}}{x-2} = \dfrac{3}{4}$.

4. Since g is not differentiable we cannot use the product rule. $f'(0) = \lim\limits_{h\to 0} \dfrac{hg(h)}{h} = 8$.

5. (b) $f'(4) = \lim\limits_{h\to 0} \dfrac{\sqrt{5-(x+h)}-1}{h} = -0.5$.

6. $F'(0) = \lim\limits_{h\to 0} \dfrac{\frac{f(h)\sin^2 h}{h}}{h} = \lim\limits_{h\to 0} \dfrac{f(h)\sin^2 h}{h^2} = f(0)$.

7. $m = e, b = 0$. Solve $\lim\limits_{x\to 1^-} e^x = \lim\limits_{x\to 1^+}(mx+b)$ and $\lim\limits_{x\to 1^-} \dfrac{e^x - e}{x-1} = \lim\limits_{x\to 1^+} \dfrac{mx+b-(m+b)}{x-1}$ for m and b.

8. (a) $S'(3) = \dfrac{F'(3)G(3) - F(3)G'(3)}{[G(3)]^2} = -\dfrac{1}{4}$. (b) $T'(0) = F'(G(0))\cdot G'(0) = 0$. (c) $U'(3) = \dfrac{F'(3)}{F(3)} = -\dfrac{1}{2}$.

9. From $h(1) = f(1)g(1)$ and $h'(1) = f'(1)g(1) + f(1)g'(1)$ it follows that $g'(1) = 9$.

10. $2f(g(1)) \cdot f'(g(1)) \cdot g'(1) = 120$.

11. $f'(x) = -\dfrac{2}{(x-2)^2}$

12. (a) $f'(x) = \sec^2 x$. This follows from $\tan x = \dfrac{\sin x}{\cos x}$ by using the quotient rule.

 (b) From $g(x) = \arctan x$, $x \in \mathbb{R}$, and $f'(g(x)) \cdot g'(x) = 1$, we conclude that $g'(x) = \cos^2(g(x))$. Next, suppose that $x > 0$ and consider the right triangle with the hypotenuse of the length 1 and with one angle measured $g(x)$ radians. Then $\tan g(x) = \tan(\arctan x) = x = \dfrac{\sin g(x)}{\cos g(x)} = \sqrt{\dfrac{1-g'(x)}{g'(x)}}$ which implies that $x^2 = \dfrac{1-g'(x)}{g'(x)}$. Thus $g'(x) = \dfrac{1}{1+x^2}$.

 (c) From $g'(x) = 2x\sec x^2 + \dfrac{2x}{1+x^4}$ it follows that $g'\left(\dfrac{\sqrt{\pi}}{2}\right) = 2\sqrt{pi} + \dfrac{16\sqrt{\pi}}{16+\pi^2}$.

13. $f'(1) = g(1) = 2$.

14. $\dfrac{d}{dx}(\sqrt{x} + x^7)\Big|_{x=1} = \dfrac{15}{2}$.

15. $f'(0) = 0$. Note that, for $h \neq 0$, $\left|\dfrac{h^2 \sin \frac{1}{h}}{h}\right| = \left|h \sin \dfrac{1}{h}\right| \leq |h|$. Use the Squeeze Theorem.

16. $f'(x) = 2 - \sin x > 0$ for all $x \in \mathbb{R}$. Let $g(0) = \alpha$. Then $g'(0) = \dfrac{1}{f'(g(0))} = \dfrac{1}{2-\sin\alpha}$.

17. Let $f(x) = \sin x$, $x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$. Then, for $x \in (-1,1)$, $(f^{-1})'(x) = \dfrac{1}{\cos(f^{-1}(x))}$. Suppose that $x \in (0,1)$ and let $\alpha = f^{-1}(x)$. Consider the right triangle with the hypothenuse of the length 1 and an angle measured α radians. The length of the leg opposite to the angle α equals $\sin\alpha = x$ which implies $\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1-x^2}}$.

18. Use the chain rule and the given property of $f'(x)$ to get $(1 + (f(g(x)))^2) \cdot g'(x) = 1$.

19. Write $y = \dfrac{1}{2} \cdot (2x^2 - 2\sqrt{x^4 - 1})$.

20. 0. Note $f(x) = (x+2)(x^2+4)$.

21. $y' = -\dfrac{5!}{x^6} - 2^5 \sin 2x$.

22. $A = 0$, $B = 1$.

23. Use the chain rule to differentiate $f^2 - g^2$.

24. Use the product rule twice.

25. $y' = \left(\dfrac{3\ln(x+2)}{x} + \dfrac{3\ln x}{x+2} - \dfrac{x}{x^2+1}\right) \cdot \dfrac{(x+2)^{3\ln x}}{(x^2+1)^{1/2}}$. Use the logarithmic differentiation.

26. $y = \dfrac{2}{\sqrt{x}} \sinh\sqrt{x} \cdot e^{4\cosh\sqrt{x}}$.

27. From $f'(x) = \dfrac{2x+1}{\sqrt{1-(x^2+x)^2}} + 5^x \ln 5$ it follows that $f'(0) = 1 + \ln 5$.

28. 0. Write as a product.

29. (a) $\dfrac{3e^{-3t}\sinh(e^{-3t})}{\sqrt{1-\cosh^2(e^{-3t})}}$

(b) $y'(u) = \dfrac{1}{3}\left(\dfrac{1}{u+1} + \dfrac{1}{u+2} - \dfrac{2u}{u^2+1} - \dfrac{2u}{u^2+2}\right) \cdot \left(\dfrac{(u+1)(u+2)}{(u^2+1)(u^2+2)}\right)^{1/3}$.

30. $y' = \dfrac{x(2\ln x+1)\sinh(\arcsin(x^2\ln x))}{\sqrt{1-x^4\ln^2 x}}$.

31. $y' = -\dfrac{4e^{-4x}}{\sqrt{1-e^{8x}}}$.

32. (a) $y' = -2xe^{\cos x^2}\sin x^2$.

(b) $y' = x^{19}\left(20\arctan x + \dfrac{x}{1+x^2}\right)$.

(c) $y' = 2x^{\ln x-1}\ln x$.

33. (a) $y' = \dfrac{6e^{3\ln(2x+1)}}{2x+1}$.

(b) $y' = 2x^{2x}(\ln x+1)$.

(c) $y' = \dfrac{e^{2x}}{(x^2+1)^3(1+\sin x)^5} \cdot \left(2 - \dfrac{6x}{x^2+1} - \dfrac{5\cos x}{1+\sin x}\right)$.

(d) $y' = \dfrac{2x+2y^2}{3-4xy}$.

34. (a) $y' = (x\cosh x + \sinh x)x^{\sinh x}$.

(b) $y' = \dfrac{xy+y^2-1}{3xy^2+3y^3-x^2-xy-1}$.

35. (a) $y' = \sec(\sinh x)\tan(\sinh x)\cosh x$.

(b) $y' = \dfrac{ex^{e-1}-e^x}{e^y-ey^{e-1}}$.

36. (a) $f'(x) = (6x-3x^2-1)e^{-x}$.

(b) $g'(z) = \dfrac{z\cos\sqrt{z^2+1}}{\sqrt{z^2+1}}$.

(c) $f'(x) = x^x(\ln x+1)$.

(d) $h'(y) = -\dfrac{y\tan y+1}{2y}\sqrt{\dfrac{\cos y}{y}}$.

37. (a) $f'(x) = \dfrac{1-x^2}{(x^2+1)^2}$.

(b) $g'(x) = \dfrac{x}{x^2+1} + 4\cot x$.

38. (a) $f'(x) = \dfrac{1}{2\sqrt{x}(1+x)}$.

(b) $f'(x) = \dfrac{5\sinh(5\ln x)}{x}$.

39. (a) $f'(x) = 3\cdot 10^{3x}\cdot\ln 10$.

(b) $f'(x) = x^9(10\tanh x + x\ \mathrm{sech}^2 x)$.

(c) $f'(x) = \left(\dfrac{\cos x}{x} - \sin x\ln x\right)x^{\cos x}$.

40. (a) $f'(x) = (2\ln x+1)x^{x^2+1}$.

(b) $f'(x) = -3\tan 3x$.

41. (a) $f'(x) = \dfrac{(5-x)(x-1)}{(x+1)^4}$.

(b) $f'(x) = 2^{2x+1}\ln 2 - \frac{4x}{3\sqrt[3]{x^2+1}}$.

(c) $f'(x) = 4x\tan(x^2)\cdot\sec^2 x^2$.

(d) $f'(x) = \left(\dfrac{\ln x}{1+x^2} + \dfrac{\arctan x}{x}\right)\cdot x^{\arctan x}$.

(e) $f'''(x) = 8\cosh(2x)$.

42. (a) $f'(x) = 5 + 5x^4 + 5^x\ln 5 + \dfrac{1}{5\sqrt[5]{x^4}}$.

(b) $y' = x^9(10\tanh x + x\ \mathrm{sech}^2 x)$.

(c) $y' = \left(\dfrac{\cos x}{x\ln x} - \sin x\ln\ln x\right)\cdot(\ln x)^{\cos x}$.

43. (a) $f'(x) = \coth x$.

(b) $f'(x) = (\cos x - x\sin x)e^{x\cos x}$.

(c) $f'(x) = \dfrac{1+\cos x}{(1+\cos x)^2}$.

(d) $f'(x) = (\ln x + 1)x^x$.

44. (a) $f'(x) = \dfrac{3}{x^4}$.

(b) $f'(x) = 2x\sin^2(2x^2) + 8x^3\sin(2x^2)\cos(x^2)$.

(c) $f'(x) = \left(\ln(x+2) + \dfrac{x}{x+2}\right)(x+2)^x$.

45. (a) $y' = \dfrac{x\sec\sqrt{x^2+1}\tan\sqrt{x^2+1}}{\sqrt{x^2+1}}$.

(b) $y' = \left(e^x\ln x + \dfrac{e^x}{x}\right)x^{e^x}$.

46. (a) $y' = 3x^2 + 3^x\ln 3 + 3(\ln x + 1)x^{3x}$.

(b) $y' = -(e^{-2x} + 4e^{-8x})$.

(c) $y' = \dfrac{y+x}{x-y}$.

(d) $y' = \left(\dfrac{5}{x} + 3x^2 + \dfrac{2x}{3(x^2+1)} - \dfrac{4}{x+1}\right)\cdot \dfrac{x^5e^{x^3}\sqrt[3]{x^2+1}}{(x+1)^4}$.

47. (a) $f'(x) = \dfrac{1}{\sec(x^2+7x)}\cdot\left(\dfrac{2x-3}{x^2-3x+8} - (2x+7)\cdot \ln(x^2-3x+8)\tan(x^2+7x)\right)$.

(b) $f'(x) = \dfrac{2\sinh(2x-3)}{1+\cosh^2(2x-3)}$

(c) $f'(x) = -3e^{3x-4}\sin(e^{3x-4})$.

(d) $f'(x) = \left((x^{-1}+2x)\ln\tan x + \dfrac{\ln x + x^2}{\sin x\cos x}\right)\cdot (\tan x)^{\ln x + x^2}$.

(e) $f'(x) = 0$.

48. (a) $h'(t) = -\dfrac{1}{3}\sec^2\left(\dfrac{t}{3}\right)\cdot e^{-\tan\left(\frac{t}{3}\right)}$

(b) $y' = -\dfrac{3}{8x\ln^2 x}\left(\dfrac{1}{2\ln x}\right)^{-1/4}$.

Note that $y = \left(\dfrac{1}{2\ln x}\right)^{3/4}$.

(c) $f'(y) = \dfrac{\ln 3}{\ln 7\cdot\sqrt{1-y^2}\cdot\arcsin y}\cdot 3^{\log_7(\arcsin y)}$

49. (a) $f'(x) = \dfrac{2x+1}{\sqrt{1-(x^2+x)^2}} + 5^x\ln 5$.

(b) $g'(x) = -\dfrac{3x^2+4x+3}{2\sqrt{x+1}(x^2-3)^2}\sinh\left(\dfrac{\sqrt{x+1}}{x^2-3}\right)$.

50. (a) $f'(x) = \dfrac{2^x(e^{4x}+a)\ln 2 - 4e^{4x}\sinh^{-1}(2^x)\sqrt{2^{2x}+1}}{(e^{4x}+a)^2\sqrt{2^{2x}+1}}$.

(b) $g'(x) = g(x)\cdot\left(-\dfrac{6x\sin(3x^2)}{2+\cos(3x^2)} + \pi - \dfrac{3}{2}\right)$. (Use logarithmic differentiation.)

51. (a) $\dfrac{d^2y}{dx^2} = -\dfrac{4x^3}{(1+x^4)^2}$

(b) $y' = \dfrac{1}{2}x^{\sqrt{x}-\frac{1}{2}}\ln(e^2x)$

6.5 Related Rates

1. Let $x = x(t)$ be the distance between the bottom of the ladder and the wall. It is given that, at any time t, $\frac{dx}{dt} = 2$ ft/s. Let $\theta = \theta(t)$ be the angle between the top of the ladder and the wall. Then $\sin\theta = \frac{x(t)}{15}$. It follows that $\cos\theta \cdot \frac{d\theta}{dt} = \frac{1}{15}\frac{dx}{dt}$. Thus when $\theta = \frac{\pi}{3}$ the rate of change of θ is given by $\frac{d\theta}{dt} = \frac{4}{15}$ ft/s.

2. Let $x = x(t)$ be the distance between the foot of the ladder and the wall and let $y = y(t)$ be the distance between the top of the ladder and the ground. It is given that, at any time t, $\frac{dx}{dt} = \frac{1}{2}$ m/min. From $x^2 + y^2 = 144$ it follows that $x \cdot \frac{dx}{dt} + y\frac{dy}{dt} = 0$. Thus when $x(t) = 4$ we have that $y(t) = 8\sqrt{2}$ and $4 \cdot \frac{1}{2} + 8\sqrt{2}\frac{dy}{dt} = 0$. The top of the ladder is falling at the rate $\frac{dy}{dt} = -\frac{\sqrt{2}}{8}$ m/min.

3. Let $x = x(t)$ be the hight of the rocket at time t and let $y = y(t)$ be the distance between the rocket and radar station. It is given that, at any time t, $x^2 = y^2 - 16$. Thus, at any time t, $x \cdot \frac{dx}{dt} = y\frac{dy}{dt}$. At the instant when $y = 5$ miles and $\frac{dy}{dt} = 3600$ mi/h we have that $x = 3$ miles and we conclude that, at that instant, $3\frac{dx}{dt} = 5 \cdot 3600$. Thus the vertical speed of the rocket is $v = \frac{dx}{dt} = 6000$ mi/h.

4. Let $x = x(t)$ be the distance between the dock and the bow of the boat at time t and let $y = y(t)$ be the length of the rope between the pulley and the bow at time t. It is given that $\frac{dy}{dt} = 1$ m/sec. From $x^2 + 1 = y^2$ it follows that $\frac{dx}{dt} = \frac{y}{x}$ m/sec. Since $y = 10$ implies $x = \sqrt{99}$ we conclude that when 10 m of rope is out then the boat is approaching the dock at the rate of $\frac{10}{\sqrt{99}}$ m/sec.

5. From $y = 5\tan\theta$ we get that, at any time t, $\frac{dy}{dt} = 5\sec^2\theta\frac{d\theta}{dt}$. At the instant when $\theta = \frac{\pi}{3}$ radians we have that $v = \frac{dy}{dt} = 5 \cdot \sec^2\frac{\pi}{3} \cdot 2 = 40$ m/s.

6. After time t (in hours) the plane is $480t$ km away from the point directly above the observer. Thus, at time t, the distance between the observer and the plane is $D = \sqrt{3^2 + (480t)^2}$. We differentiate $D^2 = 9 + 230,400t^2$ with respect to t to get $2D\frac{dD}{dt} = 460,800t$. Since 30 sec $= \frac{1}{120}$ hours it follows that the distance between the observer and the plane after 30 seconds equals $D = 5$ km. Thus, 30 seconds later the distance D from the observer to the airplane is increasing at the rate of

$$\left.\frac{dD}{dt}\right|_{t=\frac{1}{120}} = 384 \text{ km/h.}$$

7. Let y be the distance between the airplane and the radar station. Then, as the hypothenuse in a right angle triangle with the angle θ and the opposite leg of length 1000 m, $y = \frac{1000}{\sin\theta}$. Since it is given that $\frac{d\theta}{dt} = -0.1$ rad/sec, it follows that $\frac{dy}{dt} = -\frac{1000\cos\theta}{\sin^2\theta}\cdot\frac{d\theta}{dt} = \frac{100\cos\theta}{\sin^2\theta}$ m/sec. Hence if $\theta = \frac{\pi}{4}$, the speed of the plane is given by $\left.\frac{dy}{dt}\right|_{t=\frac{\pi}{4}} = 100\sqrt{2}$ m/sec.

8. (a) From $z^2 = 64 + 4t^2$ it follows that $2zz' = 8t$. If $z = 10$ then $t = 3$ and at that instant $z' = 1.2$ m/s. (b) Since the height of the kite after t seconds is $2t$ meters, it follows that $\tan x = \frac{2t}{8}$. Thus $\frac{x'}{\cos^2 x} = \frac{1}{4}$. If $y = 6$ then $t = 3$ and $\tan x = \frac{3}{4}$. It follows that $\cos x = \frac{4}{5}$ and at that instant the rate of change of x is given by $x' = x'(3) = \frac{4}{25}$ m/s.

9. Let $x = x(t)$ be the distance (in metres) between the boy and the balloon at time t. Then $[x(t)]^2 = (8t)^2 + (36+4t)^2$. From $2x(t)x'(t) = 128t + 8(36+4t)$. From $x(3) = 24\sqrt{5}$ m, it follows that $x'(3) = \frac{16}{\sqrt{5}}$ m/sec.

10. Let $\theta = \theta(t)$ be the elevation angle. From $\tan\theta = \frac{2t}{80}$ it follows that $\frac{d\theta}{dt} = \frac{\cos^2\theta}{40}$. When $t = 30$ we have $\tan\theta = \frac{3}{4}$ and $\cos\theta = \frac{4}{5}$. Thus when the helicopter is 60 m above the ground the elevation angle of the observer's line of sight to the helicopter is changing at the rate $\frac{1}{50}$ m/s.

11. Let r denotes the radius of the circular containment area. It is given that $\frac{dr}{dt} = -5$ m/min. From the fact that the area at time t is given by $A = r^2\pi$, where $r = r(t)$, it follows that $\frac{dA}{dt} = 2r\pi\frac{dr}{dt} = -10r\pi$ m^2/min. Hence when $r = 50$m then the area shrinks at the rate of $10\cdot 50\cdot\pi = 500\pi$ m^2/min.

12. Let $x = x(t)$ be the edge length. Then the volume is given by $V = x^3$ and the surface area is given by $S = 6x^2$. It is given that $\frac{dV}{dt} = 10$. This implies that $3x^2\frac{dx}{dt} = 10$ at any time t and we conclude that at the instant when $x = 8$ the edge is increasing at the rate $\frac{5}{96}$cm/min. This fact together with $\frac{dS}{dt} = 12x\frac{dx}{dt}$ implies

that at the instant when $x = 8$ the surface area is increasing at the rate $12 \cdot 8 \cdot \frac{5}{96} = 5$ cm^2/min.

13. Let $H = H(t)$ be the height of the box, let $x = x(t)$ be the length of a side of the base, and $V = V(t) = Hx^2$. It is given that $\frac{dH}{dt} = 2$ m/s and $\frac{dV}{dt} = 2x\frac{dx}{dt}H + x^2\frac{dH}{dt} = -5$ m^3/s. The question is to find the value of $\frac{dx}{dt}$ at the instant when $x^2 = 64$ m^2 and $H = 8$ m. Thus, at that instant, one of the sides of the base is decreasing at the rate of $\frac{133}{128}$ m/s.

14. Let $H = H(t)$ be the height of the pile, let $r = r(t)$ be the radius of the base, and let $V = V(t)$ be the volume of the cone. It is given that $H = r$ (which implies that $V = \frac{H^3\pi}{3}$) and that $\frac{dV}{dt} = H^2\pi\frac{dH}{dt} = 1$ m^3/sec. The question is to find the value of $\frac{dH}{dt}$ at the instant when $H = 2$. Thus at that instant the sandpile is rising at the rate of $\frac{1}{4\pi}$ m/sec.

15. Let $H = H(t)$ be the height of water, let $r = r(t)$ be the radius of the surface of water, and let $V = V(t)$ be the volume of water in the cone at time t. It is given that $r = \frac{3H}{5}$ which implies that $V = \frac{3H^3\pi}{25}$. The question is to find the value of $\frac{dH}{dt}$ at the instant when $H = 3$ and $\frac{dV}{dt} = -2$ m^3/sec . Thus at that instant the water level dropping at the rate of $\frac{50}{81\pi}$ m/sec.

16. The distance between the boy and the girl is given by $z = \sqrt{x^2 + y^2}$ where $x = x(t)$ and $y = y(t)$ are the distances covered by the boy and the girl in time t, respectively. The question is to find $z'(6)$. We differentiate $z^2 = x^2 + y^2$ to get $zz' = xx' - yy'$. From $x(6) = 9$, $y(6) = 12$, $z(6) = 15$, $x'(t) = 1.5$, and $y'(t) = 2$ it follows that $z'(6) = 2.5$ m/s.

17. The distance between the two ships is given by $z = \sqrt{x^2 + (60 - y)^2}$ where $x = x(t)$ and $y = y(t)$ are the distances covered by the ship A and the ship B in time t, respectively. The question is to find $z'(4)$. We differentiate $z^2 = x^2 + (60 - y)^2$ to get $zz' = xx' - (60 - y)y'$. From $x(4) = 60$, $y(4) = 49$, $z(4) = 61$, $x'(t) = 15$, and $y'(t) = 12.25$ it follows that $z'(4) = \frac{765.25}{61} \approx 12.54$ miles/hour.

18. Let the point L represents the lighthouse, let at time t the light beam shines on the point $A = A(t)$ on the shoreline, and let $x = x(t)$ be the distance between A and P. Let $\theta = \theta(t)$ be the measure in radians of $\angle PLA$. It is given that $x = 3\tan\theta$ and $\frac{d\theta}{dt} = 8\pi$ radians/minute. The question is to find $\frac{dx}{dt}$ at the instant when $x = 1$.

First we note that $\frac{dx}{dt} = 3\sec^2\theta \frac{d\theta}{dt}$. Secondly, at the instant when $x = 1$ we have that $\tan\theta = \frac{1}{3}$ which implies that $\cos\theta = \frac{3}{\sqrt{10}}$. Hence, when shining on a point one kilometer away from P, the light beam moving along the shoreline at the rate of $\frac{80\pi}{3}$ km/min.

19. Let $x = x(t)$ be the distance between the police car and the intersection and let $y = y(t)$ be the distance between the SUV and the intersecetion. The distance between the two cars is given by $z = \sqrt{x^2 + y^2}$. The question is to find the value of $\frac{dy}{dt}$ at the instant when $x = 0.6$ km, $y = 0.8$ km, $\frac{dz}{dt} = 20$ km/hr, and $\frac{dx}{dt} = -60$ km/hr. We differentiate $z^2 = x^2 + y^2$ to get $z\frac{dz}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$. Since, at the given instance, $z = 1$, we have that $\frac{dy}{dt} = 70$ km/hr.

6.6 Tangent Lines and Implicit Differentiation

1. Solve $y' = \cosh x = 1$. The point is $(0, 0)$.
2. Solutions of $-a^3 = 3a^2(4 - a)$ are $a = 0$ and $a = 6$. The points are $(0, 0)$ and $(6, 216)$.
3. (a) $-\frac{\pi}{4}$, (b) $y = \sqrt{2}x + 1 - \frac{\pi}{4}$.
4. $y = x - 1$
5. We note that $y' = 3(x - 1)^2$. Two lines, none of them horizontal, are perpendicular to each other if the product of their slopes equals -1. Thus to find all points on the curve C with the property that the tangent line is perpendicular to the line L we solve the equation $-\frac{1}{3} \cdot 3(x - 1)^2 = -1$. Hence $x = 0$ or $x = 2$. The lines are $y = 3x - 1$ and $y = 3x - 5$.
6. From $e^y \cdot \left(\frac{dy}{dx} \cdot \ln(x + y) + \frac{1 + \frac{dy}{dx}}{x + y} \right) = -\left(y + \frac{dy}{dx} \right) \cdot \sin(xy)$ it follows that $\left.\frac{dy}{dx}\right|_{x=1} = -1$.

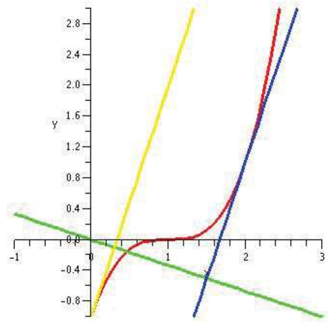

Figure 6.4: $y = (x-1)^3$ and $3y = x$

7. $\dfrac{dy}{dx} = \dfrac{y - x^4}{y^4 - x}$

8. $y \ln x = x \ln y$
$\dfrac{dy}{dx} \cdot \ln x + \dfrac{y}{x} = \ln y + \dfrac{x}{y} \cdot \dfrac{dy}{dx}$
$\dfrac{dy}{dx} = \dfrac{y(x \ln y - y)}{x(y \ln x - x)}$

9. $\dfrac{dy}{dx} = \dfrac{3^x \ln 3 + \sinh y}{e^y - x \cosh y}$

10. $\dfrac{dy}{dx} = \dfrac{\cosh x - 2xy}{x^2 - \sin y}$

11. $\dfrac{dy}{dx} = \dfrac{1 - y(x-y)}{1 + (x-y)(x+3y^2)}$

12. (a) $x + y = 0$; (b) The graph crosses the x-axis at the points $(\pm\sqrt{3}, 0)$. The claim follows from the fact that $2x - y - xy' + 2yy' = 0$ implies that if $x = \pm\sqrt{3}$ and $y = 0$ then $y' = 2$.

13. $x + y = \pi$.

14. $y = 0$.

15. $y'(3) = \dfrac{10}{21}$.

16. $\dfrac{4}{3}$

17. (a) $\dfrac{dy}{dx} = -\dfrac{2xy}{x^2 + 2ay}$; (b) We solve the system of equations $1 + a = b$, $-\dfrac{2}{1+2a} = -\dfrac{4}{3}$ to get $a = \dfrac{1}{4}$ and $b = \dfrac{5}{4}$.

18. From $\dfrac{dy}{dx} = -\sqrt{\dfrac{y}{x}}$ we get that the tangent line l to the curve at any of its points (a, b) is given by $y - b = -\sqrt{\frac{b}{a}}(x - a)$. The sum of the x-intercept and the y-intercept of l is given by $(a + \sqrt{ab}) + (b + \sqrt{ab}) = (\sqrt{a} + \sqrt{b})^2 = k$.

19. From $\dfrac{2}{3\sqrt[3]{x}} + \dfrac{2y'}{3\sqrt[3]{y}} = 0$ we conclude that $y' = -\sqrt[3]{\dfrac{y}{x}}$. Thus the tangent line through the point (a, b) on the curve is given by $y - b = -\sqrt[3]{\dfrac{b}{a}}(x - a)$. Its x and y intercepts

are $\left(a+\sqrt[3]{ab^2},0\right)$ and $\left(0,b+\sqrt[3]{a^2b}\right)$. Thus the square of the portion of the tangent line cut off by the coordinate axis is $\left(a+\sqrt[3]{ab^2}\right)^2+\left(b+\sqrt[3]{a^2b}\right)^2=a^2+2a\sqrt[3]{ab^2}+b\sqrt[3]{a^2b}+b^2+2b\sqrt[3]{a^2b}+a\sqrt[3]{ab^2}=\left(\sqrt[3]{a^2}+\sqrt[3]{b^2}\right)^3=9^3$. The length of the portion is $\sqrt{9^3}=27$.

20. $y+4=\frac{3}{4}(x-8)$.

21. (a) $(0,0)$, $(0,\pm 2)$. (b) $y'=\frac{x(2x^2-5)}{2y(y^2-2)}$. (c) $x=\sqrt{5}$.

22. (a) $y=3x-9$; (b) $y(2.98)\approx 3\cdot 2.98-9=-0.06$

23. (a) $y'(4)=4$, $y"(4)=-11$; (b) $y(3.95)\approx -0.2$; (c) Since the curve is concave down, the tangent line is above the curve and the approximation is an overestimate.

6.7 Curve Sketching

1. (a) From $f'(x)=12x^2(x-2)$ we conclude that $f'(x)>0$ for $x>2$ and $f'(x)<0$ for $x<2$. So f is increasing on $(2,\infty)$ and decreasing on $(-\infty,2)$.

   (b) From $f"(x)=12x(3x-4)$ it follows that $f"(x)>0$ for $x<0$ or $x>\frac{4}{3}$ and $f"(x)<0$ for $x\in\left(0,\frac{4}{3}\right)$. Also $f"(x)=0$ for $x=0$ and $x=\frac{4}{3}$. Thus f is concave upward on $(-\infty,0)$ and on $\left(\frac{4}{3},\infty\right)$ and concave downward on $\left(0,\frac{4}{3}\right)$

   (c) Critical numbers are $x=0$ and $x=2$. Since $f'(x)$ does not change sign at $x=0$ there is no local maximum or minimum there. (Note also that $f"(0)=0$ and that the second derivative test is inconclusive.) Since $f'(x)$ changes from negative to positive at $x=2$ there is a local minimum at $x=2$. (Note also that $f"(2)>0$, so second derivative test says there is a local minimum.)

   (d) Inflection points are $(0,10)$ and $\left(\frac{4}{3},f\left(\frac{4}{3}\right)\right)$.

   (e) $\lim_{x\to\pm\infty}f(x)=\infty$.

   For the graph see Figure 6.5.

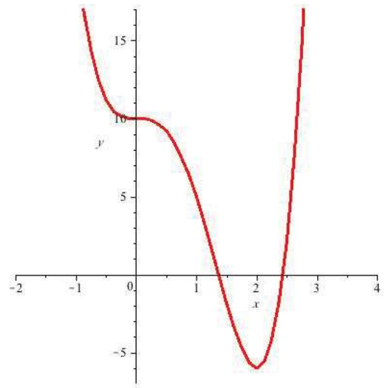

Figure 6.5: $f(x) = 3x^4 - 8x^3 + 10$

2. (a) From $x^2 - 9 > 0$ it follows that the domain of the function f is the set $(-\infty, -3) \cup (3, \infty)$. (b) The function is not defined at $x = 0$, so there is no the y-intercept. Note that $f(x) \neq 0$ for all x in the domain of f. (c) From $\lim_{x\to\infty} f(x) = 1$ and $\lim_{x\to-\infty} f(x) = -1$ we conclude that there are two horizontal asymptotes, $y = 1$ (when $x \to \infty$) and $y = -1$ (when $x \to -\infty$). From $\lim_{x\to 3^+} f(x) = 0$ and $\lim_{x\to -3^-} f(x) = -\infty$ it follows that there is a vertical asymptote at $x = -3$. (d) Since, for all x in the domain of f, $f'(x) = \dfrac{3(x-3)}{(x^2-9)^{3/2}} \neq 0$ we conclude that there is no critical number for the function f. (e) Note that $f'(x) > 0$ for $x > 3$ and $f'(x) < 0$ for $x < -3$. Thus f increasing on $(3, \infty)$ and decreasing on $(-\infty, -3)$. (f) Since the domain of f is the union of two open intervals and since the function is monotone on each of those intervals, it follows that the function f has neither (local or absolute) a maximum nor a minimum. (g) From $f''(x) = -\dfrac{6(x-3)(x-\frac{3}{2})}{(x^2-9)^{5/2}}$ it follows that $f''(x) < 0$ for all x in the domain of f. Therefore $f(x)$ is concave downwards on its domain. For the graph see Figure 6.6.

3. (a) The domain of the function f is the set $\mathbb{R}\backslash\{0\}$. The x-intercepts are ± 1. Since 0 not in domain of f there is no y-intercept. (b) From $\lim_{x\to 0^-} f(x) = -\infty$ and $\lim_{x\to 0^+} f(x) = \infty$ it follows that the vertical asymptote is the line $x = 0$. Since $\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \left(x - \dfrac{1}{x}\right) = \pm\infty$ we conclude that there is no horizontal asymptote. Finally, the fact $f(x) = x - \dfrac{1}{x}$ implies that f has the slant (oblique) asymptote $y = x$. (c) For all $x \in \mathbb{R}\backslash\{0\}$, $f'(x) = \dfrac{x^2+1}{x^2} > 0$ so the function f is increasing on

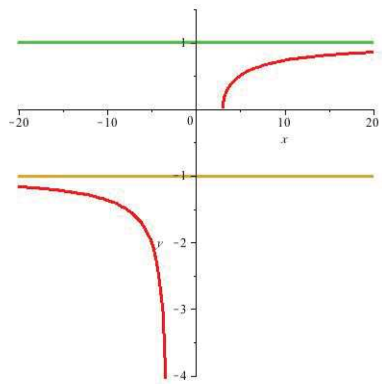

Figure 6.6: $f(x) = \dfrac{x-3}{\sqrt{x^2-9}}$

$(-\infty, 0)$ and on $(0, \infty)$. The function f has no critical numbers and thus cannot have a local maximum or minimum. (d) Since $f''(x) = -\dfrac{2}{x^3}$ it follows that $f''(x) > 0$ for $x < 0$ and $f''(x) < 0$ for $x > 0$. Therefore f is concave upward on $(-\infty, 0)$ and concave downward on $(0, \infty)$. There are no points of inflection. (e) See Figure 6.7.

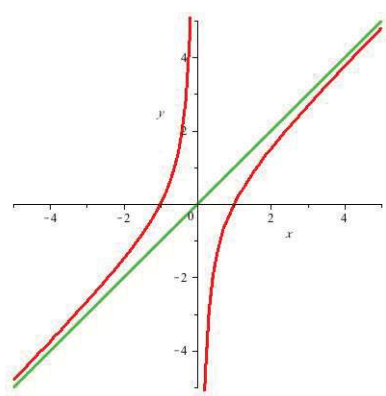

Figure 6.7: $f(x) = \dfrac{x^2-1}{x}$

4. $f(x) = x^3 - 2x^2 - x + 1$, $f'(x) = 3x^2 - 4x - 1$, $f''(x) = 6x - 4$. See Figure 6.8.

5. See Figure 6.9.

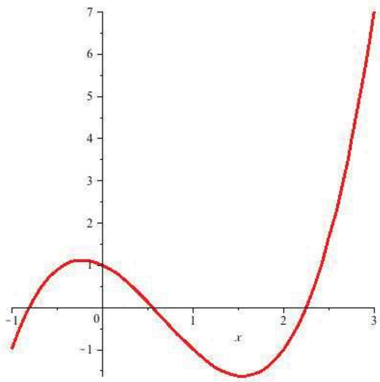

Figure 6.8: $f(x) = x^3 - 2x^2 - x + 1$ on the interval $[-1, 3]$

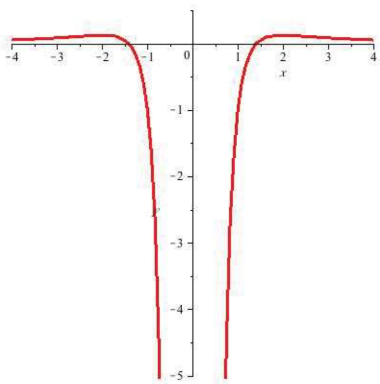

Figure 6.9: $f(x) = \dfrac{x^2 - 2}{x^4}$

6. See Figure 6.10.

7. See Figure 6.11.

8. See Figure 6.12.

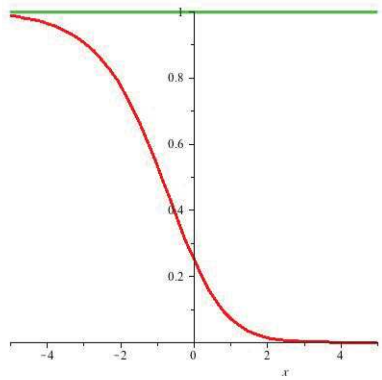

Figure 6.10: $f(x) = \dfrac{1}{(1+e^x)^2}$

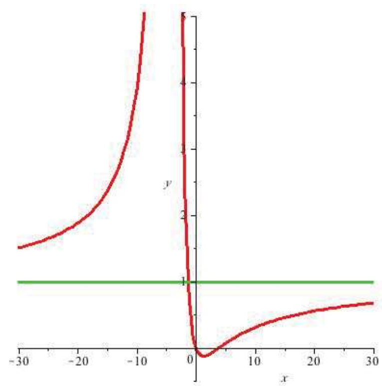

Figure 6.11: $f(x) = \dfrac{x^2-4x}{(x+4)^2}$

9. Note that the domain of the given function is the set of all real numbers. The y-intercept is the point $(0,0)$ and the x-intercepts are $(-4,0)$ and $(0,0)$. From $y' = \dfrac{4}{3}x^{1/3}\left(\dfrac{1}{x}+1\right)$ we conclude that y' is not defined at $x=0$ and that $y'=0$ if $x=-1$. Thus the critical numbers are $x=-1$ and $x=0$. Also $y'<0$ on $(-\infty,-1)$ and $y'>0$ on $(-1,0)\cup(0,\infty)$. Hence the function has a local minimum at $x=-1$. Note that the y-axis is a vertical asymptote to the graph of the given function. From $y'' = \dfrac{4}{9}x^{-5/3}(x-2)$ it follows that $y''(x)>0$ on $(-\infty,0)\cup(2,\infty$ and $y''(x)<0$ on $(0,2)$. Points of inflection are $(0,0)$ and $(2, 6\cdot 2^{1/3})$. See Figure 6.13.

10. Note that the given function is a product of a power function $y=x^{2/3}$ and a linear function $y=\dfrac{5}{2}-x$ that are both continuous on $\mathbb{R}$. See Figure 6.14.

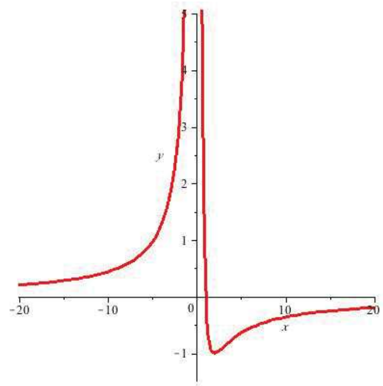

Figure 6.12: $f(x) = \dfrac{4-4x}{x^2}$

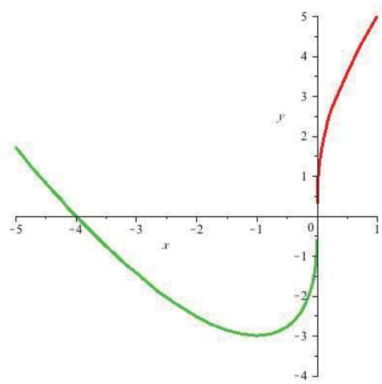

Figure 6.13: $y = 4x^{1/3} + x^{4/3}$

11. The domain is the interval $(0, \infty)$. Note that $\lim_{x\to 0^+} x^x = 1$ and $\lim_{x\to\infty} x^x = \infty$. From $y' = x^x(\ln x + 1)$ we get that the critical number is $x = \dfrac{1}{e}$. By the first derivative test there is a local minimum there. Also, $y'' = x^x[(\ln x + 1)^2 + \frac{1}{x}]$. See Figure 6.15.

12. See Figure 6.16.

13. (a) $(0, 0)$, $(3, 9e^2)$; (b) Increasing on $(-\infty, 3)$ and decreasing on $(3, \infty)$. A local (global) maximum at $(3, 9e^2)$. The other critical point is neither local maximum

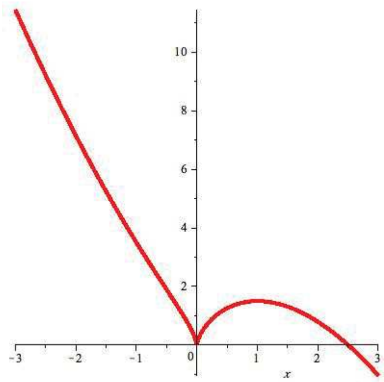

Figure 6.14: $y = x^{2/3}(\frac{5}{2} - x)$

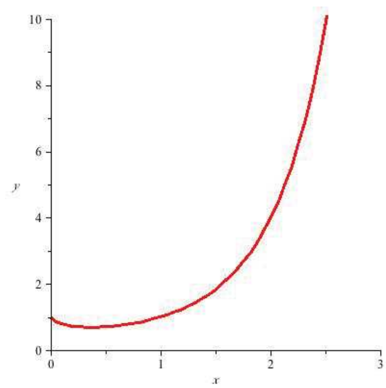

Figure 6.15: $y = x^x$

nor local minimum. (c) Note that $x^2 - 6x + 6 = (x - (3 - \sqrt{3}))(x - (3 + \sqrt{3}))$. The function is concave up on $(0, 3 - \sqrt{3})$ and $(3 + \sqrt{3}, \infty)$ and concave down on $(-\infty, 0)$ and $(3 - \sqrt{3}, 3 + \sqrt{3})$. The inflection points are $(0, 0)$, $(3 - \sqrt{3}, (3 - \sqrt{3})^3 e^{2+\sqrt{3}})$, and $(3 + \sqrt{3}, (3 + \sqrt{3})^3 e^{2-\sqrt{3}})$. (d) $\lim_{x\to-\infty} f(x) = -\infty$, $\lim_{x\to\infty} f(x) = 0$. (e) See Figure 6.17.

14. See Figure 6.18.

15. Note $\lim_{x\to 0^-} f(x) = 0$, $\lim_{x\to 0^+} f(x) = \infty$ and $\lim_{x\to\pm\infty} f(x) = 1$. See Figure 6.19.

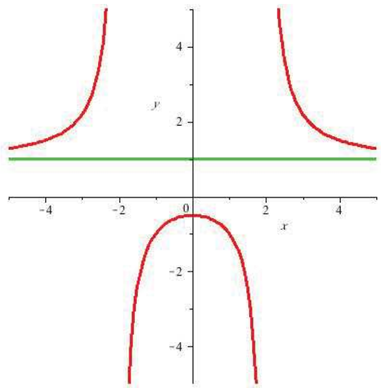

Figure 6.16: $f(x) = \dfrac{x^2+2}{x^2-4}$

16. (a) y=0. (b) f is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$. (c) Local maximum at $x = 0$. (d) Concave up on $(-\infty, -2)$ and $(2, \infty)$, concave down on $(-2, 2)$. (e) Inflection points at $x = \pm 2$.

17. See Figure 6.20.

18. See Figure 6.21.

19. Note that the function is defined on $\mathbb{R}$, but that the domain of its derivative is $\mathbb{R}\backslash\{0\}$. See Figure 6.22.

20. See Figure 6.23.

21. See Figure 6.24.

22. It is given that the y-intercept is the point $(0, -3)$. Note that the given function has a vertical asymptote $x = 3$ and two horizonatal asymtotes, $y = -1$, when $x \to -\infty$, and $y = 2$, when $x \to \infty$. Also, the function f is decreasing on $(-\infty, 3)$ and $(3, \infty)$. Finally, f is concave upwards on $(3, \infty)$ and concave downwards on $(-\infty, 3)$. See Figure 6.25.

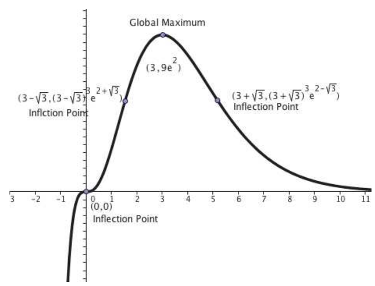

Figure 6.17: $f(x) = x^3e^{-x+5}$

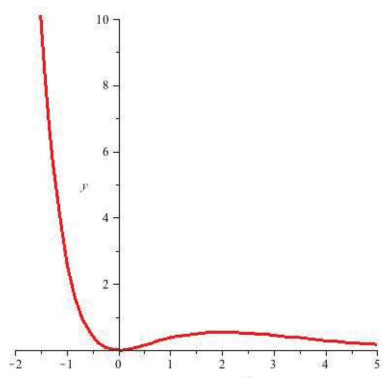

Figure 6.18: $f(x) = x^2e^{-x}$

23. (a) The graph has a vertical asymptote $y = 0$ and a horizontal asymptote $x = -2$. The following table summarizes the rest of the given information.

Interval	$(-4,-1)$	$(-1,0)$	$(0,2)$	$(2,4)$	$(4,\infty)$
Monotonity	Decreasing	Increasing	Increasing	Decreasing	Decreasing
Concavity	Downwards	Upwards	Downwards	Downwards	Upwards

(b) There are two points of inflection, $x = -1$ and $x = 4$. We note that $x = -1$ is also a critical number and that by the first derivative test there is a local minimum at $x = -1$. If $f"(-1) = 0$, then $f'(-1)$ exists and $f'(-1) = 0$. This would imply that at this point the graph of f is **above** the tangent line at $x = -1$ which contradicts

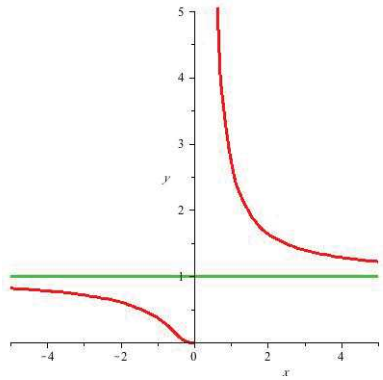

Figure 6.19: $f(x) = e^{\frac{1}{x}}$

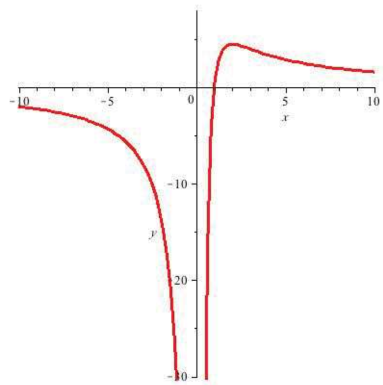

Figure 6.20: $f(x) = \dfrac{18(x-1)}{x^2}$

the fact that the curve **crosses** its tangent line at each inflection point. It follows that $f'(-1)$ does not exist and therefore $f"(-1)$ does not exist.

For a graph see Figure 6.26.

24. (a) $\surd$: r, s, g; (b) $\surd$: r, s, f, g; (c) $\surd$: r; (d) $\surd$: g; (e) $\surd$: r.

25. (a) $\surd$: C, D; (b) $\surd$: A; (c) $\surd$: A, D; (d) $\surd$: A, B, C, D; (e) $\surd$: B.

26. $a = -3$, $b = 7$. Solve the system $y(1) = a + b + 2 = 6$ and $y"(1) = 6 + 2a = 0$.

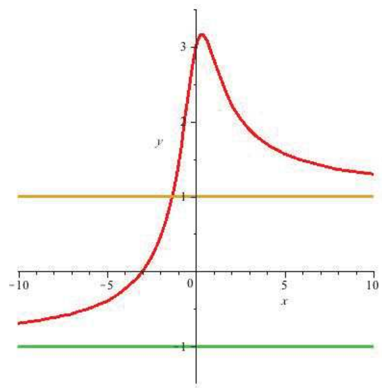

Figure 6.21: $f(x) = \dfrac{x+3}{\sqrt{x^2+1}}$

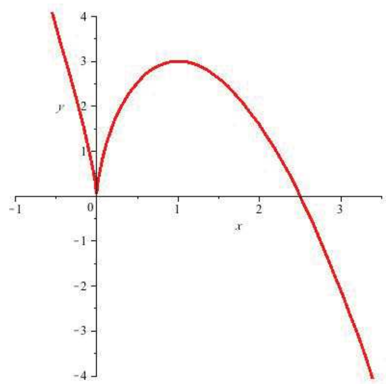

Figure 6.22: $f(x) = (5-2x)x^{\frac{2}{3}}$

6.8 Optimization

1. Note that the function f is continuous on the closed interval $[-1, 2]$. By the Intermediate Value Theorem the function f attains its maximum and minimum values on $[-1, 2]$. To find those global extrema we evaluate and compare the values of f at the endpoints and critical numbers that belong to $(-1, 2)$. From $f'(x) = 6x - 9 = 3(2x-3)$ we conclude that the critical number is $x = \dfrac{3}{2}$. From $f(-1) = 12$, $f(2) = -6$, and $f\left(\dfrac{3}{2}\right) = -\dfrac{27}{4}$ we conclude that the maximum value is $f(-1) = 12$ and the minimum value is $f\left(\dfrac{3}{2}\right) = -\dfrac{27}{4}$.

2. The global minimum value is $f(-4) = f(2) = -21$, and the global maximum value is $f(6) = 139$. Note that $f(2) = -21$ is also a local minimum and that $f(-2)$ is a

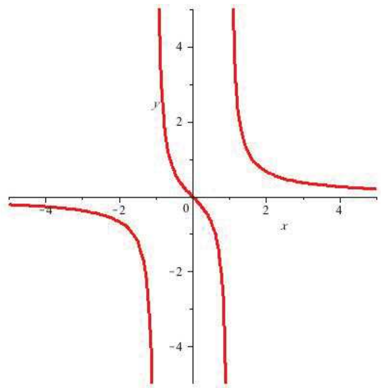

Figure 6.23: $f(x) = \dfrac{x}{x^2-1}$

local maximum. (**Reminder:** By our definition, for $x = c$ to be a local extremum of a function f it is necessary that c is an interior point of the domain of f. This means that there is an open interval I contained in the domain of f such that $c \in I$.)

3. From $f'(x) = ax^{a-1}(1-x)^b - bx^a(1-x)^{b-1} = x^{a-1}(1-x)^{b-1}(a-(a+b)x)$ and the fact that a and b are positive conclude that $x = \dfrac{a}{a+b} \in (0,1)$ is a critical number of the function f. Since $f(0) = f(1) = 0$ and $f(x) > 0$ for all $x \in (0,1)$ it follows that the maximum value of f is $f\left(\dfrac{a}{a+b}\right) = \left(\dfrac{a}{a+b}\right)^a \left(\dfrac{b}{a+b}\right)^b$.

4. From $f(x) = \begin{cases} 3x-5 & \text{if } x \geq \frac{5}{3} \\ -3x+5 & \text{if } x < \frac{5}{3} \end{cases}$ we conclude that $f'(x) = \begin{cases} 3 & \text{if } x > \frac{5}{3} \\ -3 & \text{if } x < \frac{5}{3} \end{cases}$. Thus, for $x \neq \dfrac{5}{3}$, $f'(x) \neq 0$ and the derivative of f is not defined at $x = \dfrac{5}{3}$. We conclude that the only critical number of the function f on the interval $[-3, 2]$ is $x = \dfrac{5}{3}$. Clearly, $f\left(\dfrac{5}{3}\right) = 0$. From $f(-3) = 14$ and $f(1) = 2$ it follows that the global and local minimum is $f\left(\dfrac{5}{3}\right) = 0$ and that the global maximum is $f(-3) = 14$.

5. The question is to find the minimum value of the function $f(x) = x^2 + (12-x)^2$, $x \in (0, 12)$. From $f'(x) = 4(x-6)$ it follows that $x = 6$ is the only critical number. From $f"(6) = 4 > 0$, by the second derivative test, it follows that $f(6) = 72$ is the minimum value of the function f.

6. Note that $f(0) = f(1) = 0$ and that $f(x) > 0$ for $x \in (0,1)$. Thus by the Intermediate Value Theorem there is $c \in (0,1)$ such that $f(c)$ is the maximum value of f. Since f is differentiable on $(0,1)$, c must be a critical point. Note that $f'(x) = x^{a-1}(1-x)^{b-1}(a-(a+b)x)$. Since a and b are both positive we have that

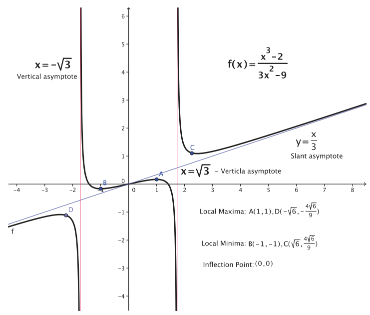

Figure 6.24: $f(x) = \dfrac{x^3 - 2x}{3x^2 - 9}$

$x = \dfrac{a}{a+b} \in (0, 1)$. Thus $x = \dfrac{a}{a+b}$ is the only critical point of the function f in the interval $(0, 1)$ and $f\left(\dfrac{a}{a+b}\right) = \dfrac{a^a b^b}{(a+b)^{a+b}}$ is the maximum value.

7. The distance between a point (x, y) on the curve and the point $(0, -3)$ is $d = \sqrt{(x-0)^2 + (y-(-3))^2} = \sqrt{y^4 + (y+3)^2}$. The question is to minimize the function $f(y) = y^4 + (y+3)^2$, $y \in \mathbb{R}$. From $f'(y) = 2(2y^3 + y + 3) = 2(y+1)(2y^2 - 2y + 3)$we conclude that $y = -1$ is the only critical number of the function f. From $f"(-1) = 10 > 0$, by the second derivative test we conclude that $f(-1) = 5$ is the (local and global) minimum value of f. Thus the closest point is $(-1, -1)$.

8. Let x be the radius of the circle. The question is to minimize the function $f(x) = \pi x^2 + \left(\dfrac{40 - 2\pi x}{4}\right)^2$, $x \in \left(0, \dfrac{20}{\pi}\right)$. (We are given that there are TWO pieces.) The only critical number of the function f is $x = \dfrac{20}{\pi + 4}$. To minimize the total area the two pieces should be of the length $\dfrac{40\pi}{\pi+4}$ and $\dfrac{160}{\pi+4}$.

9. $x = 3$ and $y = 2$. The question is to minimize the function $f(x) = x^2 + \dfrac{3(7-x)^2}{4}$, $x \in (0, 7)$. (We are given that there are TWO pieces.) The critical point is $x = 3$.

10. The question is to **maximize** the function $f(x) = \dfrac{x^2\sqrt{3}}{36} + \dfrac{(4-x)^2}{16}$, $x \in [0, 4]$. Note that $f"(x) > 0$ for $x \in (0, 4)$ and conclude that the maximum value must occur at

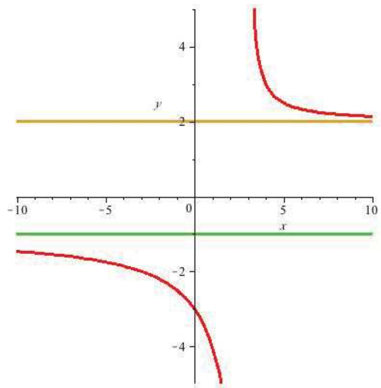

Figure 6.25: Two Horizontal Asymptotes

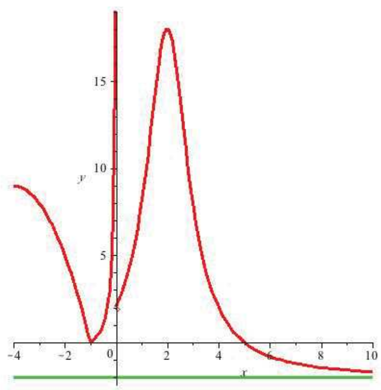

Figure 6.26: Discontinuity at $x = 0$

$x = 0$ and/or $x = 4$. Since $f(4) < f(0)$, the maximum total area is obtained if only the square is constructed.

11. A rectangle with sides parallel to the coordinate axes is to be inscribed in the region enclosed by the graphs of $y = x^2$ and $y = 4$ so that its perimeter has maximum length.

    (a) See Figure 6.27.

    (b) $P = 4a + 2(4 - a^2) = 2(4 + 2a - a^2)$, $a \in (0, 1)$.

    (c) From $\dfrac{dP}{da} = 4(1 - a)$ it follows that $a = 1$ is the only critical number. The fact that $f"(a) = -8 < 0$ for all $a \in (0, 1)$ implies, by the second derivative test, that $P(1)$ is the maximum value.

    (d) $P(1) = 10$.

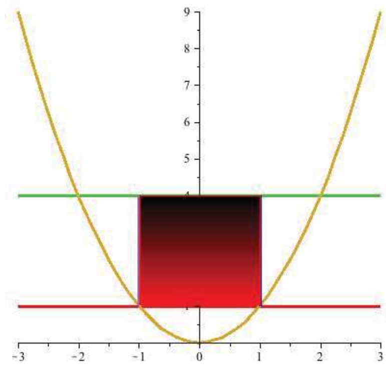

Figure 6.27: Rectangle insribed in the region $y = 4$ and $y = x^2$

12. Let $(x, 0)$ be the bottom right vertex of the rectangle. The question is to maximize $f(x) = 2x(12 - x^2)$, $x \in (0, 2\sqrt{3})$. The only critical number is $x = 2$. The length of the rectangle with the largest area is 4 and its height is 8.

13. Let x be the length of one side of the fence that is perpendicular to the wall. Note that the length of the side of the fence that is parallel to the wall equals $400 - 2x$ and that this number cannot be larger than 100.The question is to maximize the function $f(x) = x(400 - 2x)$, $x \in [150, 400)$. The only solution of the equation $f'(x) = 4(100 - x) = 0$ is $x = 100$ but this value is not in the domain of the function f. Clearly $f'(x) < 0$ for $x \in [150, 400)$ which implies that f is decreasing on its domain. Therefore the maximum area that can be enclosed is $f(150) = 15000$ ft^2.

14. $L = 15\sqrt{3}$. To minimize $L^2 = (x + 5)^2 + y^2$, use the fact that $\dfrac{x}{10\sqrt{2}} = \dfrac{x + 5}{y}$ and the first derivative.

15. Let $(x, y) = \left(x, \dfrac{b}{a}\sqrt{a^2 - x^2}\right)$ be the upper right vertex of the rectangle. The question is to maximize the function $f(x) = \dfrac{4b}{a}x\sqrt{a^2 - x^2}$, $x \in (0, a)$. From $f'(x) = \dfrac{4b}{a}\dfrac{a^2 - 2x^2}{\sqrt{a^2 - x^2}}$ we conclude that the only critical number is $x = \dfrac{a}{\sqrt{2}}$. By the first derivative test, there is a local maximum at this critical number. Since $\lim_{x \to 0^+} f(x) = \lim_{x \to a^-} f(x) = 0$, it follows that $f\left(\dfrac{a}{\sqrt{2}}\right) = 2ab$ is the maximum value of the function f. Thus to maximize the area of the soccer field its length should be $a\sqrt{2}$ and its width should be $b\sqrt{2}$.

16. Let a be the length of the printed material on the poster. Then the width of this area equals $b = \dfrac{384}{a}$. It follows that the length of the poster is $x = a + 8$ and the

width of the poster is $y = b + 12 = \dfrac{384}{a} + 12$. The question is to minimize the function $f(a) = xy = (a+8)\left(\dfrac{384}{a} + 12\right) = 12\left(40 + a + \dfrac{256}{a}\right)$. It follows that the function has a local minimum at $a = 16$. The dimensions of the poster with the smallest area are $x = 24$ cm and $y = 36$ cm.

17. $(\sqrt{15} + 2) \times (2\sqrt{15} + 4)$.

18. Let P be the point on the shore where Maya lands her boat and let x be the distance from P to the point on the shore that is closest to her initial position. Thus to reach the village she needs to row the distance $z = \sqrt{4 + x^2}$ and run the distance $y = 6 - x$. Time needed to row the distance z is given by $T_1 = \dfrac{z}{2}$ and time she needs to run is $T_2 = \dfrac{y}{5}$. Therefore the question is to minimize the function $T = T(x) = T_1 + T_2 = \dfrac{\sqrt{4 + x^2}}{2} + \dfrac{6 - x}{5}$, $x \in [0, 6]$. From $f'(x) = \dfrac{x}{2\sqrt{4 + x^2}} - \dfrac{1}{5}$ it follows that the only critical number is $x = \dfrac{4}{3}$. From $T(0) = \dfrac{11}{5} = 2.2$, $T(6) = \sqrt{10}$, and $T\left(\dfrac{4}{3}\right) \approx 2.135183758$ it follows that the minimum value is $T\left(\dfrac{4}{3}\right)$. Maya should land her boat $\dfrac{4}{3}$ km from the point initially nearest to the boat.

19. (a) Let y be the height of the box. Then the surface area is given by $S = 2x^2 + 4xy$. From $S = 150$ it follows that $y = \dfrac{1}{2}\left(\dfrac{75}{x} - x\right)$. Therefore the volume of the box is given by $V = V(x) = \dfrac{x}{2}\left(75 - x^2\right)$.

    (b) From the fact that $y = \dfrac{1}{2}\left(\dfrac{75}{x} - x\right) > 0$ it follows that the domain of the function $V = V(x)$ is the interval $[1, 5\sqrt{3})$.

    (c) Note that $\dfrac{dV}{dx} = \dfrac{3}{2}(25 - x^2)$ and that $\dfrac{d^2V}{dx^2} = -3x < 0$ for all $x \in (1, 5\sqrt{3})$. Thus the maximum value is $V(5) = 125$ cube units.

20. (a) Note that $y = \dfrac{10}{x^2}$. The cost function is given by $C(x) = 5x^2 + 2 \cdot 4 \cdot x \cdot \dfrac{10}{x^2} = 5x^2 + \dfrac{80}{x}$, $x > 0$.

    (b) $2 \times 2 \times \dfrac{5}{2}$. The minimum cost is $C(2) = \$60$.

21. Let x be the length and the width of the box. Then its height is given by $y = \dfrac{13500}{x^2}$. It follows that the surface area is $S = x^2 + \dfrac{54000}{x}$ cm^2, $x > 0$. The question is to

minimize S. From $\dfrac{dS}{dx} = 2x - \dfrac{54000}{x^2}$ and $\dfrac{d^2S}{dx^2} = 2 + \dfrac{3 \cdot 54000}{x^3} > 0$ for all $x > 0$ it follows that the function S has a local and global minimum at $x = 30$.

22. We need to maximize the area of the trapezoid with parallel sides of lengths $a = 2$ and $c = 2 + 2 \cdot 2\cos\theta = 2 + 4\cos\theta$ and the height $h = 2\sin\theta$. Thus we maximize the function $A = A(\theta) = \dfrac{2 + (2 + 4\cos\theta)}{2} \cdot 2\sin\theta = 4(\sin\theta + \sin\theta\cos\theta)$, $\theta \in (0, \pi)$. From $\dfrac{dA}{d\theta} = 4(\cos\theta + \cos^2\theta - \sin^2\theta) = 4(2\cos^2\theta + \cos\theta - 1) = 4(2\cos\theta - 1)(\cos\theta + 1)$ we obtain the critical number $\theta = \dfrac{\pi}{3}$. The First Derivative Test confirms that $\theta = \dfrac{\pi}{3}$ maximizes the cross sectional area of the trough.

23. Let r be the radius of the base of a cylinder inscribed in the cone and let h be its height. From $\dfrac{H}{R} = \dfrac{h}{R - r}$ (see Figure 6.28)

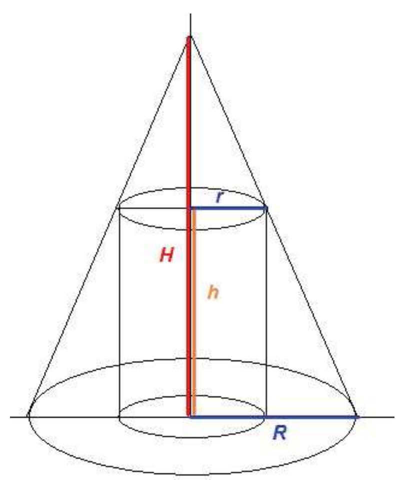

Figure 6.28: Cylinder insribed in a cone

we conclude that $h = \dfrac{H(R - r)}{R}$. Thus the volume of the cylinder is $V = V(r) = \dfrac{\pi H}{R} r^2(R - r)$, $r \in (0, R)$. From $\dfrac{dV}{dr} = \dfrac{\pi H}{R} r(2R - 3r)$ and $\dfrac{d^2V}{dr^2} = \dfrac{2\pi H}{R}(R - 3r)$ it follows that the maximum value of the volume of the cylinder is $V\left(\dfrac{2R}{3}\right) = \dfrac{4\pi H R^2}{27}$. The dimensions are $r = \dfrac{2R}{3}$ and $h = \dfrac{H}{3}$.

24. $r = \sqrt{\dfrac{10}{3\pi}}$ m, $h = \left(\dfrac{5}{\pi}\sqrt{\dfrac{3\pi}{10}} - \dfrac{1}{2}\sqrt{\dfrac{10}{3\pi}}\right)$ m; $V = \dfrac{10}{3}\sqrt{\dfrac{10}{3\pi}}$ m^3.

25. Let r be the radius of the base of the pot. Then the height of the pot is $h = \dfrac{250}{\pi r^2}$. The cost function is $C(r) = 4\pi r^2 + \dfrac{1000}{r}$, $r > 0$. The cost function has its minimum at $r = \dfrac{5}{\sqrt[3]{\pi}}$.

26. (a) The surface area of the can is $S = 2\pi rh + 2\pi r^2$. The amount of material wasted is $A - S = 2(4 - \pi)r^2$.

    (b) From $V = \pi r^2 h$ it follows that the amount of material needed to make a can of the given volume V is $A = A(r) = \dfrac{2V}{r} + 8r^2$. This function has its minimum at $r = \dfrac{\sqrt[3]{V}}{2}$. The ratio of the height to diameter for the most economical can is $\dfrac{h}{r} = \dfrac{4}{\pi}$.

    (c) $A''(r) = \dfrac{4V}{r^3} + 8 > 0$ for $r > 0$.

27. $r = \sqrt[3]{20}$, $h = \dfrac{14}{\sqrt[3]{50}}$. Minimize the cost function $C = C(r) = 7r^2\pi + \dfrac{280\pi}{r}$.

28. From Figure 6.29 conclude that $r^2 = R^2 - (h - R)^2 = h(4R - h)$. Then the volume of the cone as a function of h is given by $V = \dfrac{\pi}{3}h^2(4R - h)$. Maximize.

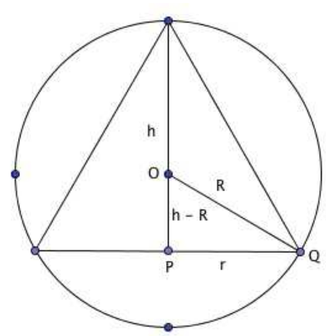

Figure 6.29: Cone insrribed in a sphere

29. Let P be the source power of the first party's stereo and let x be the distance between the person and the first party. Since the power of the second party's stereo is $64P$, the sound level is $L(x) = kPx^{-2} + 64kP(100 - x)^{-2}$, $x \in (0, 100)$. From

$ds\frac{dL}{dx} = 2kP\left(\frac{64}{(100-x)^3} - \frac{1}{x^3}\right)$ it follows that $x = 20$ is the only critical number for the function L. Since for $x \in (0, 100)$

$$L'(x) > 0 \Leftrightarrow \frac{64}{(100-x)^3} - \frac{1}{x^3} > 0 \Leftrightarrow 64x^3 > (100-x)^3 \Leftrightarrow 4x > 100 - x \Leftrightarrow x > 20$$

the function L is strictly increasing on the interval $(20, 100)$ and strictly decreasing on the interval $(0, 20)$. Therefore, $L(20)$ is the absolute minimum.

6.9 Mean Value Theorem

1. Since $x + 7 \neq 0$ for all $x \in [-1, 2]$ it follows that the function g, as a rational function, is continuous on the closed interval $[-1, 2]$ and differentiable on the open interval $(-1, 2)$. Therefore the function g satisfies he hypothesis of the Mean Value Theorem on the interval $[-1, 2]$. By the Mean Value Theorem there is $c \in (-1, 2)$ such that $g'(c) = \dfrac{g(2) - g(-1)}{2 - (-1)}$. Thus the question is to solve $\dfrac{21}{(c+7)^2} = \dfrac{7}{18}$ for c. Hence $c = -7 \pm 3\sqrt{6}$. Clearly $-7 - 3\sqrt{6} < -1$ and this value is rejected. From

$$-7 + 3\sqrt{6} > -1 \Leftrightarrow 3\sqrt{6} > 6 \text{ and } -7 + 3\sqrt{6} < 2 \Leftrightarrow 3\sqrt{6} < 9$$

it follows that $c = -7 + 3\sqrt{6} \in (-1, 2)$ and it is the only value that satisfies the conclusion of the Mean Value Theorem.

2. The inequality is obviously satisfied if $a = b$. Let $a, b \in \mathbb{R}$, $a < b$, and let $f(x) = \sin x$, $x \in [a, b]$. Clearly the function f is continuous on the closed interval $[a, b]$ and differentiable on (a, b). Thus, by the Mean value Theorem, there is $c \in (a, b)$ such that $\cos c = \dfrac{\sin b - \sin a}{b - a}$. Since $|\cos c| \leq 1$ for all real numbers c it follows that $|\sin b - \sin a| \leq |b - a|$.

3. Let $f(t)$ be the distance that the first horse covers from the start in time t and let $g(t)$ be the distance that the second horse covers from the start in time t. Let T be time in which the two horses finish the race. It is given that $f(0) = g(0)$ and $f(T) = g(T)$. Let $F(t) = f(t) - g(t)$, $t \in [0, T]$. As the difference of two position functions, the function F is continuous on the closed interval $[0, T]$ and differentiable on the open interval $(0, T)$. By the Mean value Theorem there is $c \in (0, T)$ such that $F'(c) = \dfrac{F(T) - F(0)}{T - 0} = 0$. It follows that $f'(c) = g'(c)$ which is the same as to say that at the instant c the two horse have the same speed. (Note: It is also possible to use Rolle's theorem.)

4. (a) $[a,b]$; (a,b); $f'(c)(b-a)$; (b) Note that all conditions of the Mean Value Theorem are satisfied. To get the bounds use the fact that, for some $c \in (1,3)$, $f(5) - f(3) = 2f'(c)$. (c) Note that $h(2) - h(0) = 0$ and apply the Mean Value theorem for the function h on the closed interval $[0,2]$.

6.10 Differential, Linear Approximation, Newton's Method

1. (a) Note that $f(0) = 8$. From $f'(x) = \frac{3}{2}\sqrt{x+4}$ it folows that $f'(0) = 3$. Thus the linearization of f at $a = 0$ is $L(x) = 8 + 3x$.

   (b) For x "close" to 0 we have that $f(x) = \frac{3}{2}\sqrt{(x+4)^3} \approx L(x)$. Thus $\sqrt{(3.95)^3} = f(-0.05) \approx L(-0.05) = 8 - 0.15 = 7.85$. Since $f"(x) = \frac{3}{4\sqrt{x+4}} > 0$ we conclude that, in the neighborhood of $x = 0$, the graph of the function f is above the tangent line at $x = 0$. Thus $L(-0.05)$ is an underestimate.

2. Let $f(x) = x^{\frac{2}{3}}$. Then $f(x) = \frac{2}{3}x^{-\frac{1}{3}}$, $f(27) = 9$, and $f'(27) = \frac{2}{9}$. Hence the linearization of the function f at $a = 27$ is $L(x) = 9 + \frac{2}{9}(x - 27)$. It follows that $\sqrt[3]{26^2} = f(26) \approx L(26) = 9 - \frac{2}{9} = \frac{79}{9}$. (Note: MAPLE gives $\frac{79}{9} \approx 8.777777778$ and $\sqrt[3]{26^2} \approx 8.776382955$.)

3. Let $f(x) = x^{\frac{2}{3}}$. Then $f(x) = \frac{2}{3}x^{-\frac{1}{3}}$, $f(64) = 16$, and $f'(64) = \frac{1}{6}$. Hence the linearization of the function f at $a = 64$ is $L(x) = 16 + \frac{1}{6}(x - 64)$. It follows that $(63)^{2/3} = f(63) \approx L(63) = 16 - \frac{1}{6} = \frac{95}{6}$. The error is close to the absolute value of the differential $|dy| = |f'(64)\Delta x| = \frac{1}{6}$. (Note: MAPLE gives $\frac{95}{6} \approx 15.83333333$ and $\sqrt[3]{63^2} \approx 15.83289626$.)

4. Let $f(x) = \sqrt{x}$. Then $f(x) = \frac{1}{2\sqrt{x}}$, $f(81) = 9$, and $f'(81) = \frac{1}{18}$. Hence the linearization of the function f at $a = 81$ is $L(x) = 9 + \frac{1}{18}(x - 81)$. It follows that $\sqrt{80} = f(80) \approx L(80) = 9 - \frac{1}{18} = \frac{161}{18}$. (Note: MAPLE gives $\frac{161}{18} \approx 8.944444444$ and $\sqrt{80} \approx 8.944271910$.)

5. The linearization of the function f at $a = 5$ is $L(x) = 2 + 4(x - 5)$. Thus $f(4.9) \approx L(4.9) = 2 - 0.4 = 1.6$.

6. (a) The linearization of the function g at $a = 2$ is $L(x) = -4 + 3(x - 2)$. Thus $g(2.05) \approx L(2.05) = -3.85$. (b) From $g"(2) = \frac{2}{3} > 0$ we conclude that the function g is concave downward at $a = 2$, i.e., the graph of the function lies below the tangent line. Thus, the estimate is larger than the actual value.

7. (a) $L(x) = 1 - \frac{x}{2}$. (b) $\sqrt{0.9} \approx 1 - \frac{9}{20} = \frac{11}{20}$. (c) $y = -\frac{x}{2} + 1$. (d) See Figure 6.30.

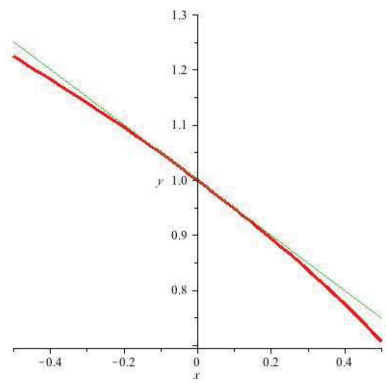

Figure 6.30: $f(x) = \sqrt{1 - x}$ and its tangent at $x = 0$

8. (a) $L(x) = 1 + x$. (b) $\sqrt{1.1} = f(0.05) \approx L(0.05) = 1.05$. (c) An over-estimate since f is concave-down. MAPLE gives $\sqrt{1.1} \approx 1.048808848$.

9. (a) $L(x) = 2 + \frac{x}{12}$. (b) $\sqrt[3]{7.95} \approx L(-0.05) = 2 - \frac{1}{240} = \frac{479}{240}$ and $\sqrt[3]{8.1} \approx L(0.1) = 2 + \frac{1}{120} = \frac{243}{120}$. (Note: MAPLE gives $\frac{479}{240} \approx 1.995833333$ and $\sqrt[3]{7.95} \approx 1.995824623$. Also, $\frac{243}{120} \approx 2.025000000$ and $\sqrt[3]{8.1} \approx 2.008298850$.)

10. (a) $y = \frac{x}{9} + 3$. (b) $\sqrt[3]{30} \approx \frac{1}{9} + 3 = \frac{28}{9}$. (Note: MAPLE gives $\frac{28}{9} \approx 3.111111111$ and $\sqrt[3]{30} \approx 3.107232506$.) (c) See Figure 6.31.

11. The linearization of the function $f(x) = \ln x$ at $a = 1$ is given by $L(x) = x - 1$. Thus $\ln 0.9 \approx L(0.9) = -0.1$. (Note: MAPLE gives $\ln 0.9 \approx -.1053605157$.)

12. (a) $L(x) = x - 1$. (b) Let $x = \exp(-0.1)$. Then $\ln x = -0.1 \approx L(x) = x - 1$. Thus $x \approx 0.9$. (Note: MAPLE gives $\exp(-0.1) \approx 0.9048374180$.)

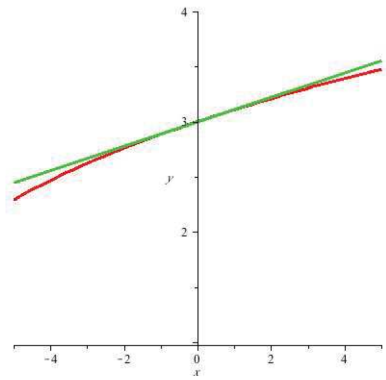

Figure 6.31: $f(x) = \sqrt[3]{27 - 3x}$ and its tangent at $x = 0$

13. $L(x) = 10 + \frac{1}{300}(x - 1000)$ implies $1001^{1/3} \approx L(1001) = \frac{3001}{300}$. (Note: MAPLE gives $\frac{3001}{300} \approx 10.00333333$ and $\sqrt[3]{1001} \approx 10.00333222$.)

14. (a) The linearization of the function $f(x) = \sqrt{x} + \sqrt[5]{x}$ at $a = 1$ is given by $L(x) = 2 + \frac{7}{10}(x - 1)$. Thus $f(1.001) \approx L(1.001) = 2 + 0.7 \cdot 0.001 = 2.0007$. (b) Note that the domain of the function f is the interval $[0, \infty)$. From $f"(x) = -\frac{1}{4}x^{-\frac{3}{2}} - \frac{4}{25}x^{-\frac{9}{5}}$ it follows that f is concave downwards on the interval $(0, \infty)$. (c) The graph of the function is below the tangent line at $a = 1$, so the estimate $f(1.001) \approx 2.0007$ is too high.

15. (a) $L(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}(x - \frac{\pi}{6})$. (b) By the Mean Value Theorem, for $x > \frac{\pi}{6}$ and some $c \in (\frac{\pi}{6}, x)$, $\frac{f(x) - f(\frac{\pi}{6})}{x - \frac{\pi}{6}} = \frac{\sin x - \frac{1}{2}}{x - \frac{\pi}{6}} = f'(c) = \cos c \leq 1$. Since $x - \frac{\pi}{6} > 0$, the inequality follows. (c) From (a) and (b) it follows that, for $x > \frac{\pi}{6}$, $\sin x \leq \frac{1}{2} + (x - \frac{\pi}{6}) < \frac{1}{2} + \frac{\sqrt{3}}{2}(x - \frac{\pi}{6}) = L(x)$. Next, $\Delta f = f(x) - f(\frac{\pi}{6}) = \sin x - \frac{1}{2} < L(x) - \frac{1}{2} = \frac{\sqrt{3}}{2}(x - \frac{\pi}{6}) = f'(\frac{\pi}{6})\Delta x = df$.

16. (b) Let $f(x) = \cos x - x^2$. Then $f'(x) = -\sin x - 2x$. Thus $x_2 = 1 - \frac{\cos 1 - 1}{-\sin 1 - 2} \approx 0.8382184099$, $x_2 = 0.8382184099 - \frac{\cos 0.8382184099 - 0.8382184099^2}{-\sin 0.8382184099 - 2 \cdot 0.8382184099} \approx 0.8242418682$, and $x_3 = 0.8242418682 - \frac{\cos 0.8242418682 - 0.8242418682^2}{-\sin 0.8242418682 - 2 \cdot 0.8242418682} \approx 0.8241323190$. (Note: MAPLE gives $\cos 0.8241323190 - 0.8241323190^2 \approx -1.59 \cdot 10^{-8}$.)

17. (b) Take $f(x) = \sqrt[3]{x}$, $x_0 = 1$, $x_1 = -2$, $x_2 = 4$, and $x_3 = -8$. See Figure 6.32.

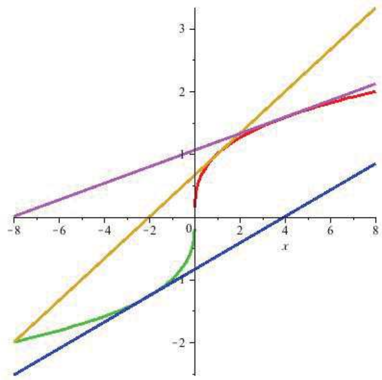

Figure 6.32: Newton's Method fails: $f(x) = \sqrt[3]{x}$ and $x_0 = 1$

18. (a) We use Newton's Method to solve the equation $x^2 - 5 = 0$, $x > 0$. From $f(x) = x^2 - 5$ and $f'(x) = 2x$, Newton's Method gives $x_{n+1} = x_n - \dfrac{x_n^2 - 5}{2x_n} = \dfrac{1}{2}\left(x_n + \dfrac{5}{x_n}\right)$.

    (b) A rough estimate of $\sqrt{5}$ gives a value that is a bit bigger than 2. Thus, take $x_1 = 1$.

    (c) $x_2 = 3$, $x_3 = \dfrac{7}{3}$, $x_4 = \frac{47}{21} \approx 2.23809$. (Note: MAPLE gives $\sqrt{5} \approx 2.23606$.)

19. Let $f(x) = x^{\frac{1}{3}}$. Then Newton's method gives $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} = x_n - \dfrac{x_n^{\frac{1}{3}}}{\frac{1}{3}x_n^{-\frac{2}{3}}} = -2x_n$. So $|x_{n+1}| = 2|x_n|$. This implies that if $x_0 \neq 0$, $|x_n| = 2^n|x_0| \to \infty$ as $n \to \infty$; Newton's Method does not work in this case! See Figure 6.32.

20. (a) Take $f(x) = x^5 - k$. Then $f'(x) = 5x^4$ and $x_{n+1} = x_n - \dfrac{x_n^5 - k}{5x_n^4} = \dfrac{4x_n^5 + k}{5x_n^4} = \dfrac{x_n}{5}\left(4 + \dfrac{k}{x_n^5}\right)$. (b) $x_{n+1} = \sqrt[5]{k}$. (c) $x_2 = 1.85$. [MAPLE gives $\sqrt[5]{20} \approx 1.820564203$.]

21. From $f(x) = x^5 - 31$ and $f'(x) = 5x^4$ it follows that $x_1 = \dfrac{159}{80}$ and $x_2 = \dfrac{159}{80} - \dfrac{\left(\frac{159}{80}\right)^5 - 31}{5 \cdot \left(\frac{159}{80}\right)^4} \approx 1.987340780$. (Note: MAPLE gives $\sqrt[5]{31} = 1.987340755$.)

22. (b) The question is approximate the solution of the equation $F(x) = \sin x - x = 0$ with $x_0 = \dfrac{\pi}{2}$. Thus $x_1 = \dfrac{\pi}{2} - \dfrac{\sin\frac{\pi}{2} - \frac{\pi}{2}}{\cos\frac{\pi}{2} - 1} = 1$. (Note: Clearly the solution of the given equation is $x = 0$. Newton's method with $x_0 = \frac{\pi}{2}$ gives $x_7 = 0.08518323251$.)

23. $x_2 = 1 - \dfrac{-1}{-22} = \dfrac{21}{22}$. (Note: MAPLE gives $\dfrac{21}{22} \approx 0.9545454545$ and approximates the solution of the equation as 0.9555894038)

24. (a) $x_1 = 2 - \dfrac{-1}{4} = \dfrac{9}{4}$. (b) The question is to approximate a solution of the equation $f'(x) = 0$ with the initial guess $x_0 = 2$, $f'(2) = 4$, and $f''(2) = 3$ given. Hence $x_1 = 2 - \frac{4}{3} = \frac{2}{3}$.

25. (a) From $f(x) = \dfrac{1}{x} - a$ and $f'(x) = -\dfrac{1}{x^2}$ it follows that $x_{n+1} = x_n - \dfrac{\frac{1}{x_n} - a}{-\frac{1}{x_n^2}} = 2x_n - ax_n^2$. (b) Note that $\dfrac{1}{1.128}$ is the solution of the equation $\dfrac{1}{x} - 1.128 = 0$. Thus $x_2 = 2 - 1.128 = 0.872$, $x_3 = 2 \cdot 0.872 - 1.128 \cdot 0.872^2 = 0.886286848$, and $x_4 = 0.8865247589$. (Note: MAPLE gives $\dfrac{1}{1.128} \approx 0.8865248227$.)

26. $x_2 = \dfrac{\pi}{2} - \dfrac{1 - \frac{\pi}{4}}{-\frac{1}{2}} = 2$. (Note: MAPLE estimates the positive solution of the equation $\sin x = \dfrac{x}{2}$ as 1.895494267. Newton's method with the initial guess $x_1 = \dfrac{\pi}{2}$ gives $x_3 \approx 1.900995594$.)

27. (a) From $f'(x) = 3(x^2 - 1)$ it follows that the critical numbers are $x = \pm 1$. From $f(1) = 3$, $f(-1) = 7$, $\lim\limits_{x\to-\infty} f(x) = -\infty$, and $\lim\limits_{x\to\infty} f(x) = \infty$ it follows that f has only one root and that root belongs to the interval $(-\infty, -1)$. From $f(-2) = 3 > 0$ and $f(-3) = -13 < 0$, by the Intermediate Value Theorem, we conclude that the root belongs to the interval $(-3, -2)$. (b)Let $x_0 = -3$. Then $x_1 = -3 - \dfrac{-13}{504} = -\dfrac{1499}{504} \approx -2.974206349$ and $x_3 \approx -2.447947724$. It seems that Newton's method is working, the new iterations are inside the interval $(-3, -2)$ where we know that the root is. (Note: MAPLE estimates the solution of the equation $x^3 - 3x + 5 = 0$ as $x = -2.279018786$.)

28. (a) The function f is continuous on the closed interval $\left[-\dfrac{1}{2}, 0\right]$ and $f\left(-\dfrac{1}{2}\right) = -\dfrac{5}{8} < 0$ and $f(0) = 1 > 0$. By the Intermediate Value Theorem, the function f has at least one root in the interval $\left(-\dfrac{1}{2}, 0\right)$. (b) Take $x_1 = -\dfrac{1}{3}$. Then $x_2 =$

$-\frac{1}{3} - \frac{-\frac{1}{27} - 3 \cdot \left(-\frac{1}{3}\right) + 1}{3\left(\frac{1}{9}+1\right)} = -\frac{29}{90} \approx -.3222222222$ and $x_3 \approx -.3221853550$. (Note: MAPLE estimates the solution of the equation $x^3+3x+1=0$ as $x = -.3221853546$.)

29. (a) Take $f(x) = \ln x + x^2 - 3$, evaluate $f(1)$ and $f(3)$, and then use the Intermediate Value Theorem. (b) Note that $f'(x) = \frac{1}{x} + 2x > 0$ for $x \in (1,3)$. (c) From $f(1) = -2$ and $f'(1) = 3$ it follows that $x_2 = \frac{5}{3} \approx 1.66$. [MAPLE gives 1.592142937 as the solution.]

30. (a) Take $f(x) = 2x - \cos x$, evaluate $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to\infty} f(x)$, and then use the Intermediate Value Theorem. (b) Note that $f'(x) = 2 + \sin x > 0$ for $x \in \mathbb{R}$. (c) From $f(0) = -1$ and $f'(0) = 2$ it follows that $x_2 = \frac{1}{2}$. [MAPLE gives 0.4501836113 as the solution.]

31. (a) Take $f(x) = 2x - 1 - \sin x$, evaluate $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to\infty} f(x)$, and then use the Intermediate Value Theorem. (b) Note that $f'(x) = 2 - \cos x > 0$ for $x \in \mathbb{R}$. (c) From $f(0) = -1$ and $f'(0) = 1$ it follows that $x_2 = 1$. [MAPLE gives 0.8878622116 as the solution.]

6.11 Antiderivatives and Differential Equations

1. $f(x) = 2\sin x + 2x^4 - e^x + 8$.

2. $g(x) = -\cos x - x^{-1} - e^x + \pi^{-1} + e^\pi$

3. $f(x) = \frac{1}{3}x^3 + x^2 + 3x - \frac{242}{3}$ and $f(1) = -\frac{229}{3}$.

4. $h(1) = 2e(1-e^2) - \frac{44}{3}$.

5. $F(z) = \frac{1}{2}\ln(z^2+9)$.

6. It is given that $f(0) = 1$ and $f'(0) = 0$. Thus $f(x) = x^3 + 1$.

7. $\int \frac{dx}{x(1+\ln x)} = \ln(1+\ln x) + C$.

8. For each case compute the indefinite integral.

   (a) $F(x) = -\frac{1}{9}(1-x)^9$

   (b) $\int \tan^2 x dx = \int (\sec^2 x - 1)dx = \tan x - x + C$

   (c) $F(x) = \frac{1}{\sqrt{2}} \arctan \frac{x}{\sqrt{2}} + C$

   (d) $F(x) = \frac{1}{6}e^{3x} + \frac{1}{2}e^x + C$

9. $f(t) = 2e^t - 3\sin t + t - 2$.

10. It is given that $x(0) = 10$, $x'(0) = v(0) = 0$ and $x''(t) = 12t$. Hence $x(t) = 2t^3 + 10$.

11. (a) $v(t) = \frac{3}{2}t^2 + 6$. (b) 4 seconds. Solve $v(t) = 30$.

12. (a) Let $s(t)$ be the height of the ball after t seconds. It is given that $s(0) = 0$, $s'(0) = v(0) = 64$ ft/sec and $s"(0) = v'(0) = a(0) = -32$ ft/sec^2. Thus $s(t) = -16t^2 + 64t = 16t(4 - t)$. From $s(4) = 0$ it follows that the ball is in the air for 4 seconds. (b) $v(4) = s'(4) = -64$ ft/sec^2.

13. (a) From the fact that the velocity of an falling object is approximated by $v(t) = -gt + v(0)$ and the fact that, in the given case, $v(t) = 0$, we conclude that the distance $y = y(t)$ between the ball and the surface of the Earth at time t is given by $\frac{dy}{dt} = -gt$. Hence, $y = -\frac{gt^2}{2} + H$, where H is the height of the blimp at the moment when the ball was dropped. At the moment when the ball hits surface we have that $0 = -\frac{gt^2}{2} + H$ which implies that it takes $t = \sqrt{\frac{2H}{g}}$ seconds for a ball to drop H meters. (b) $v = -g \cdot \sqrt{\frac{2H}{g}} = -10 \cdot 7 = -70$ m/sec.

14. $y = \frac{-2\cos 3x + x^3 + e^{3x} + 1}{3} + x$.

15. $y = \sin\left(x + \frac{\pi}{2}\right)$.
Solve $\frac{dy}{\sqrt{1-y^2}} = dx$.

16. $y = \tan\left(x - \frac{\pi}{4}\right)$.

17. $y = 4e^x - 1$.

18. $x(t) = -3(4t - 7)^{-3} + 4$

19. $y = \ln(x + e^2)$.

20. $y = 2 - e^{-t}$.

21. (a) $y = \frac{3}{2}\sin 2x - \frac{1}{4}\exp(-4x) + \frac{5}{4}$. (b) $F(x) = \frac{1}{2}\ln(2x + 1) + C$.

22. Let x be the number of towels sold per week at the price $p = p(x)$. Let $C = C(x)$ be the cost of manufacturing x towels. It is given that $\frac{dC}{dx} = 0.15$ CAN/towel and $\frac{dp}{dx} = -\frac{0.10}{50}$ CAN/towel. Hence $C(x) = 0.15x + a$ and $p(x) = -\frac{0.10x}{50} + b$, for some constants a and b (in CAN). Then the profit is given by $P = P(x) =$ Revenue − Cost $= x \cdot p(x) - C(x) = -\frac{0.10x^2}{50} + bx - 0.15x - a$. The quantity that maximizes revenue is $x = 1000$ towels and it must be a solution of the equation $\frac{dP}{dx} = -\frac{0.10x}{25} + b - 0.15 = 0$. Hence $-\frac{0.10 \cdot 1000}{25} + b - 0.15 = 0$ and $b = 4.15$ CAN. The price that maximizes the profit is $p = -\frac{0.10 \cdot 1000}{50} + 4.15 = 2.15$ CAN.