point(s) of discontinuity occur between the limits of integration or at the limits themselves. If the point of discontinuity occurs outside of the limits of integration the integral can still be evaluated.

In the following sets of examples we won't make too much of an issue with continuity problems, or lack of continuity problems, unless it affects the evaluation of the integral. Do not let this convince you that you don't need to worry about this idea. It arises often enough that it can cause real problems if you aren't on the lookout for it.

Finally, note the difference between indefinite and definite integrals. Indefinite integrals are functions while definite integrals are numbers.

Let's work some more examples.

Example 2 Evaluate each of the following.

(a) $\int_{-3}^{1} 6x^2 - 5x + 2\,dx$ [Solution]

(b) $\int_{4}^{0} \sqrt{t}\left(t-2\right)dt$ [Solution]

(c) $\displaystyle\int_{1}^{2} \frac{2w^5 - w + 3}{w^2}\,dw$ [Solution]

(d) $\int_{25}^{-10} dR$ [Solution]

Solution

(a) $\int_{-3}^{1} 6x^2 - 5x + 2\,dx$

There isn't a lot to this one other than simply doing the work.

$$\begin{aligned}\int_{-3}^{1} 6x^2 - 5x + 2\,dx &= \left(2x^3 - \frac{5}{2}x^2 + 2x\right)\Bigg|_{-3}^{1} \\ &= \left(2 - \frac{5}{2} + 2\right) - \left(-54 - \frac{45}{2} - 6\right) \\ &= 84\end{aligned}$$

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(b) $\int_{4}^{0} \sqrt{t}\left(t-2\right)dt$

Recall that we can't integrate products as a product of integrals and so we first need to multiply the integrand out before integrating, just as we did in the indefinite integral case.

$$\int_4^0 \sqrt{t}\left(t-2\right)dt = \int_4^0 t^{\frac{3}{2}} - 2t^{\frac{1}{2}}\,dt$$
$$= \left(\frac{2}{5}t^{\frac{5}{2}} - \frac{4}{3}t^{\frac{3}{2}}\right)\Bigg|_4^0$$
$$= 0 - \left(\frac{2}{5}(4)^{\frac{5}{2}} - \frac{4}{3}(4)^{\frac{3}{2}}\right)$$
$$= -\frac{32}{15}$$

In the evaluation process recall that,

$$(4)^{\frac{5}{2}} = \left((4)^{\frac{1}{2}}\right)^5 = (2)^5 = 32$$
$$(4)^{\frac{3}{2}} = \left((4)^{\frac{1}{2}}\right)^3 = (2)^3 = 8$$

Also, don't get excited about the fact that the lower limit of integration is larger than the upper limit of integration. That will happen on occasion and there is absolutely nothing wrong with this.

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(c) $\displaystyle\int_1^2 \frac{2w^5 - w + 3}{w^2}\,dw$

First, notice that we will have a division by zero issue at $w = 0$, but since this isn't in the interval of integration we won't have to worry about it.

Next again recall that we can't integrate quotients as a quotient of integrals and so the first step that we'll need to do is break up the quotient so we can integrate the function.

$$\int_1^2 \frac{2w^5 - w + 3}{w^2}\,dw = \int_1^2 2w^3 - \frac{1}{w} + 3w^{-2}\,dw$$
$$= \left(\frac{1}{2}w^4 - \ln\left|w\right| - \frac{3}{w}\right)\Bigg|_1^2$$
$$= \left(8 - \ln 2 - \frac{3}{2}\right) - \left(\frac{1}{2} - \ln 1 - 3\right)$$
$$= 9 - \ln 2$$

Don't get excited about answers that don't come down to a simple integer or fraction. Often times they won't. Also don't forget that $\ln\left(1\right) = 0$.

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(d) $\int_{25}^{-10} dR$

This one is actually pretty easy. Recall that we're just integrating 1!.

$$\begin{aligned}\int_{25}^{-10} dR &= R\Big|_{25}^{-10} \\ &= -10 - 25 \\ &= -35\end{aligned}$$

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The last set of examples dealt exclusively with integrating powers of x. Let's work a couple of examples that involve other functions.

Example 3 Evaluate each of the following.

(a) $\int_0^1 4x - 6\sqrt[3]{x^2}\, dx$ [Solution]

(b) $\int_0^{\frac{\pi}{3}} 2\sin\theta - 5\cos\theta\, d\theta$ [Solution]

(c) $\int_{\pi/6}^{\pi/4} 5 - 2\sec z \tan z\, dz$ [Solution]

(d) $\int_{-20}^{-1} \frac{3}{\mathbf{e}^{-z}} - \frac{1}{3z}\, dz$ [Solution]

(e) $\int_{-2}^{3} 5t^6 - 10t + \frac{1}{t}\, dt$ [Solution]

Solution

(a) $\int_0^1 4x - 6\sqrt[3]{x^2}\, dx$.

This one is here mostly here to contrast with the next example.

$$\begin{aligned}\int_0^1 4x - 6\sqrt[3]{x^2}\, dx &= \int_0^1 4x - 6x^{\frac{2}{3}}\, dx \\ &= \left(2x^2 - \frac{18}{5}x^{\frac{5}{3}}\right)\Bigg|_0^1 \\ &= 2 - \frac{18}{5} - (0) \\ &= -\frac{8}{5}\end{aligned}$$

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(b) $\int_0^{\frac{\pi}{3}} 2\sin\theta - 5\cos\theta\, d\theta$

Be careful with signs with this one. Recall from the indefinite integral sections that it's easy to mess up the signs when integrating sine and cosine.

$$\int_0^{\frac{\pi}{3}} 2\sin\theta - 5\cos\theta \, d\theta = \left(-2\cos\theta - 5\sin\theta\right)\Big|_0^{\pi/3}$$
$$= -2\cos\left(\frac{\pi}{3}\right) - 5\sin\left(\frac{\pi}{3}\right) - \left(-2\cos 0 - 5\sin 0\right)$$
$$= -1 - \frac{5\sqrt{3}}{2} + 2$$
$$= 1 - \frac{5\sqrt{3}}{2}$$

Compare this answer to the previous answer, especially the evaluation at zero. It's very easy to get into the habit of just writing down zero when evaluating a function at zero. This is especially a problem when many of the functions that we integrate involve only x's raised to positive integers; these evaluate is zero of course. After evaluating many of these kinds of definite integrals it's easy to get into the habit of just writing down zero when you evaluate at zero. However, there are many functions out there that aren't zero when evaluated at zero so be careful.

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(c) $\int_{\pi/6}^{\pi/4} 5 - 2\sec z \tan z \, dz$

Not much to do other than do the integral.

$$\int_{\pi/6}^{\pi/4} 5 - 2\sec z \tan z \, dz = \left(5z - 2\sec z\right)\Big|_{\pi/6}^{\pi/4}$$
$$= 5\left(\frac{\pi}{4}\right) - 2\sec\left(\frac{\pi}{4}\right) - \left(5\left(\frac{\pi}{6}\right) - 2\sec\left(\frac{\pi}{6}\right)\right)$$
$$= \frac{5\pi}{12} - 2\sqrt{2} + \frac{4}{\sqrt{3}}$$

For the evaluation, recall that

$$\sec z = \frac{1}{\cos z}$$

and so if we can evaluate cosine at these angles we can evaluate secant at these angles.

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(d) $\int_{-20}^{-1} \frac{3}{\mathbf{e}^{-z}} - \frac{1}{3z} \, dz$

In order to do this one will need to rewrite both of the terms in the integral a little as follows,

$$\int_{-20}^{-1} \frac{3}{\mathbf{e}^{-z}} - \frac{1}{3z} \, dz = \int_{-20}^{-1} 3\mathbf{e}^{z} - \frac{1}{3}\frac{1}{z} \, dz$$

For the first term recall we used the following fact about exponents.

$$x^{-a} = \frac{1}{x^a} \qquad\qquad \frac{1}{x^{-a}} = x^a$$

In the second term, taking the 3 out of the denominator will just make integrating that term easier.

Now the integral.

$$\begin{aligned}\int_{-20}^{-1} \frac{3}{\mathbf{e}^{-z}} - \frac{1}{3z}\,dz &= \left(3\mathbf{e}^{z} - \frac{1}{3}\ln|z|\right)\Bigg|_{-20}^{-1} \\ &= 3\mathbf{e}^{-1} - \frac{1}{3}\ln|-1| - \left(3\mathbf{e}^{-20} - \frac{1}{3}\ln|-20|\right) \\ &= 3\mathbf{e}^{-1} - 3\mathbf{e}^{-20} + \frac{1}{3}\ln|20|\end{aligned}$$

Just leave the answer like this. It's messy, but it's also exact.

Note that the absolute value bars on the logarithm are required here. Without them we couldn't have done the evaluation.

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(e) $\displaystyle\int_{-2}^{3} 5t^6 - 10t + \frac{1}{t}\,dt$

This integral can't be done. There is division by zero in the third term at $t = 0$ and $t = 0$ lies in the interval of integration. The fact that the first two terms can be integrated doesn't matter. If even one term in the integral can't be integrated then the whole integral can't be done.

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So, we've computed a fair number of definite integrals at this point. Remember that the vast majority of the work in computing them is first finding the indefinite integral. Once we've found that the rest is just some number crunching.

There are a couple of particularly tricky definite integrals that we need to take a look at next. Actually they are only tricky until you see how to do them, so don't get too excited about them. The first one involves integrating a piecewise function.

Example 4 Given,

$$f(x) = \begin{cases} 6 & \text{if } x > 1 \\ 3x^2 & \text{if } x \le 1 \end{cases}$$

Evaluate each of the following integrals.

(a) $\displaystyle\int_{10}^{22} f(x)\,dx$ [Solution]

(b) $\displaystyle\int_{-2}^{3} f(x)\,dx$ [Solution]

Solution

Let's first start with a graph of this function.