Integrating logarithms requires a topic that is usually taught in Calculus II and so we won't be integrating a logarithm in this class. Also note the third integrand can be written in a couple of ways and don't forget the absolute value bars in the x in the answer to the third integral.

Finally, let's take care of the inverse trig and hyperbolic functions.

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1} x + c \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + c$$

$$\int \sinh x\,dx = \cosh x + c \qquad \int \cosh x\,dx = \sinh x + c$$

$$\int \operatorname{sech}^2 x\,dx = \tanh x + c \qquad \int \operatorname{sech} x \tanh x\,dx = -\operatorname{sech} x + c$$

$$\int \operatorname{csch}^2 x\,dx = -\coth x + c \qquad \int \operatorname{csch} x \coth x\,dx = -\operatorname{csch} x + c$$

As with logarithms integrating inverse trig functions requires a topic usually taught in Calculus II and so we won't be integrating them in this class. There is also a different answer for the second integral above. Recalling that since all we are asking here is what function did we differentiate to get the integrand the second integral could also be,

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = -\cos^{-1} x + c$$

Traditionally we use the first form of this integral.

Okay, now that we've got most of the basic integrals out of the way let's do some indefinite integrals. In all of these problems remember that we can always check our answer by differentiating and making sure that we get the integrand.

Example 1 Evaluate each of the following indefinite integrals.

(a) $\int 5t^3 - 10t^{-6} + 4\,dt$ [Solution]

(b) $\int x^8 + x^{-8}\,dx$ [Solution]

(c) $\int 3\sqrt[4]{x^3} + \frac{7}{x^5} + \frac{1}{6\sqrt{x}}\,dx$ [Solution]

(d) $\int dy$ [Solution]

(e) $\int \left(w + \sqrt[3]{w}\right)\left(4 - w^2\right)dw$ [Solution]

(f) $\int \frac{4x^{10} - 2x^4 + 15x^2}{x^3}\,dx$ [Solution]

Solution

Okay, in all of these remember the basic rules of indefinite integrals. First, to integrate sums and differences all we really do is integrate the individual terms and then put the terms back together with the appropriate signs. Next, we can ignore any coefficients until we are done with integrating that particular term and then put the coefficient back in. Also, do not forget the "+*c*" at the end it is important and must be there.

So, let's evaluate some integrals.

(a) $\int 5t^3 - 10t^{-6} + 4\,dt$

There's not really a whole lot to do here other than use the first two formulas from the beginning of this section. Remember that when integrating powers (that aren't -1 of course) we just add one onto the exponent and then divide by the new exponent.

$$\begin{aligned}\int 5t^3 - 10t^{-6} + 4\,dt &= 5\left(\frac{1}{4}\right)t^4 - 10\left(\frac{1}{-5}\right)t^{-5} + 4t + c\\ &= \frac{5}{4}t^4 + 2t^{-5} + 4t + c\end{aligned}$$

Be careful when integrating negative exponents. Remember to add one onto the exponent. One of the more common mistakes that students make when integrating negative exponents is to "add one" and end up with an exponent of "-7" instead of the correct exponent of "-5".

[Return to Problems]

(b) $\int x^8 + x^{-8}\,dx$

This is here just to make sure we get the point about integrating negative exponents.

$$\int x^8 + x^{-8}\,dx = \frac{1}{9}x^9 - \frac{1}{7}x^{-7} + c$$

[Return to Problems]

(c) $\int 3\sqrt[4]{x^3} + \frac{7}{x^5} + \frac{1}{6\sqrt{x}}\,dx$

In this case there isn't a formula for explicitly dealing with radicals or rational expressions. However, just like with derivatives we can write all these terms so they are in the numerator and they all have an exponent. This should always be your first step when faced with this kind of integral just as it was when differentiating.

$$\begin{aligned}\int 3\sqrt[4]{x^3} + \frac{7}{x^5} + \frac{1}{6\sqrt{x}}\,dx &= \int 3x^{\frac{3}{4}} + 7x^{-5} + \frac{1}{6}x^{-\frac{1}{2}}\,dx\\ &= 3\frac{1}{7/4}x^{\frac{7}{4}} - \frac{7}{4}x^{-4} + \frac{1}{6}\left(\frac{1}{1/2}\right)x^{\frac{1}{2}} + c\\ &= \frac{12}{7}x^{\frac{7}{4}} - \frac{7}{4}x^{-4} + \frac{1}{3}x^{\frac{1}{2}} + c\end{aligned}$$

When dealing with fractional exponents we usually don't "divide by the new exponent". Doing this is equivalent to multiplying by the reciprocal of the new exponent and so that is what we will usually do.

[Return to Problems]

(d) $\int dy$

Don't make this one harder than it is…

$$\int dy = \int 1\,dy = y + c$$

In this case we are really just integrating a one!

[Return to Problems]

(e) $\int \left(w + \sqrt[3]{w}\right)\left(4 - w^2\right)dw$

We've got a product here and as we noted in the previous section there is no rule for dealing with products. However, in this case we don't need a rule. All that we need to do is multiply things out (taking care of the radicals at the same time of course) and then we will be able to integrate.

$$\begin{aligned}\int \left(w + \sqrt[3]{w}\right)\left(4 - w^2\right)dw &= \int 4w - w^3 + 4w^{\frac{1}{3}} - w^{\frac{7}{3}}\,dw \\ &= 2w^2 - \frac{1}{4}w^4 + 3w^{\frac{4}{3}} - \frac{3}{10}w^{\frac{10}{3}} + c\end{aligned}$$

[Return to Problems]

(f) $\int \frac{4x^{10} - 2x^4 + 15x^2}{x^3}\,dx$

As with the previous part it's not really a problem that we don't have a rule for quotients for this integral. In this case all we need to do is break up the quotient and then integrate the individual terms.

$$\begin{aligned}\int \frac{4x^{10} - 2x^4 + 15x^2}{x^3}\,dx &= \int \frac{4x^{10}}{x^3} - \frac{2x^4}{x^3} + \frac{15x^2}{x^3}\,dx \\ &= \int 4x^7 - 2x + \frac{15}{x}\,dx \\ &= \frac{1}{2}x^8 - x^2 + 15\ln\left|x\right| + c\end{aligned}$$

Be careful to not think of the third term as x to a power for the purposes of integration. Using that rule on the third term will NOT work. The third term is simply a logarithm. Also, don't get excited about the 15. The 15 is just a constant and so it can be factored out of the integral. In other words, here is what we did to integrate the third term.

$$\int \frac{15}{x}\,dx = 15\int \frac{1}{x}\,dx = 15\ln\left|x\right| + c$$

[Return to Problems]

Always remember that you can't integrate products and quotients in the same way that we integrate sums and differences. At this point the only way to integrate products and quotients is to multiply the product out or break up the quotient. Eventually we'll see some other products and quotients that can be dealt with in other ways. However, there will never be a single rule that will work for all products and there will never be a single rule that will work for all quotients. Every product and quotient is different and will need to be worked on a case by case basis.

The first set of examples focused almost exclusively on powers of x (or whatever variable we used in each example). It's time to do some examples that involve other functions.

Example 2 Evaluate each of the following integrals.

(a) $\int 3\mathbf{e}^{x} + 5\cos x - 10\sec^{2} x\,dx$ [Solution]

(b) $\int 2\sec w\tan w + \frac{1}{6w}\,dw$ [Solution]

(c) $\int \frac{23}{y^{2}+1} + 6\csc y\cot y + \frac{9}{y}\,dy$ [Solution]

(d) $\int \frac{3}{\sqrt{1-x^{2}}} + 6\sin x + 10\sinh x\,dx$ [Solution]

(e) $\int \frac{7-6\sin^{2}\theta}{\sin^{2}\theta}\,d\theta$ [Solution]

Solution

Most of the problems in this example will simply use the formulas from the beginning of this section. More complicated problems involving most of these functions will need to wait until we reach the Substitution Rule.

(a) $\int 3\mathbf{e}^{x} + 5\cos x - 10\sec^{2} x\,dx$

There isn't anything to this one other than using the formulas.

$$\int 3\mathbf{e}^{x} + 5\cos x - 10\sec^{2} x\,dx = 3\mathbf{e}^{x} + 5\sin x - 10\tan x + c$$

[Return to Problems]

(b) $\int 2\sec w\tan w + \frac{1}{6w}\,dw$

Let's be a little careful with this one. First break it up into two integrals and note the rewritten integrand on the second integral.

$$\begin{aligned}\int 2\sec w\tan w + \frac{1}{6w}\,dw &= \int 2\sec w\tan w\,dw + \int \frac{1}{6}\frac{1}{w}\,dw \\ &= \int 2\sec w\tan w\,dw + \frac{1}{6}\int \frac{1}{w}\,dw\end{aligned}$$

Rewriting the second integrand will help a little with the integration at this early stage. We can think of the 6 in the denominator as a 1/6 out in front of the term and then since this is a constant it can be factored out of the integral. The answer is then,

$$\int 2\sec w\tan w+\frac{1}{6w}\,dw=2\sec w+\frac{1}{6}\ln|w|+c$$

Note that we didn't factor the 2 out of the first integral as we factored the 1/6 out of the second. In fact, we will generally not factor the 1/6 out either in later problems. It was only done here to make sure that you could follow what we were doing.

[Return to Problems]

(c) $\displaystyle\int \frac{23}{y^2+1}+6\csc y\cot y+\frac{9}{y}\,dy$

In this one we'll just use the formulas from above and don't get excited about the coefficients. They are just multiplicative constants and so can be ignored while we integrate each term and then once we're done integrating a given term we simply put the coefficients back in.

$$\int \frac{23}{y^2+1}+6\csc y\cot y+\frac{9}{y}\,dy=23\tan^{-1}y-6\csc y+9\ln|y|+c$$

[Return to Problems]

(d) $\displaystyle\int \frac{3}{\sqrt{1-x^2}}+6\sin x+10\sinh x\,dx$

Again, there really isn't a whole lot to do with this one other than to use the appropriate formula from above.

$$\int \frac{3}{\sqrt{1-x^2}}+6\sin x+10\sinh x\,dx=3\sin^{-1}x-6\cos x+10\cosh x+c$$

[Return to Problems]

(e) $\displaystyle\int \frac{7-6\sin^2\theta}{\sin^2\theta}\,d\theta$

This one can be a little tricky if you aren't ready for it. As discussed previously, at this point the only way we have of dealing with quotients is to break it up.

$$\begin{aligned}\int \frac{7-6\sin^2\theta}{\sin^2\theta}\,d\theta&=\int \frac{7}{\sin^2\theta}-6\,d\theta\\&=\int 7\csc^2\theta-6\,d\theta\end{aligned}$$

Notice that upon breaking the integral up we further simplified the integrand by recalling the definition of cosecant. With this simplification we can do the integral.

$$\int \frac{7-6\sin^2\theta}{\sin^2\theta}\,d\theta=-7\cot\theta-6\theta+c$$

[Return to Problems]