Unit 3

Exponential and Logarithmic Functions

Exponential and logarithmic functions can be used to describe and solve a wide range of problems. Some of the questions that can be answered using these two types of functions include:

- How much will your bank deposit be worth in five years, if it is compounded monthly?
- How will your car loan payment change if you pay it off in three years instead of four?
- How acidic is a water sample with a pH of 8.2?
- How long will a medication stay in your bloodstream with a concentration that allows it to be effective?
- How thick should the walls of a spacecraft be in order to protect the crew from harmful radiation?

In this unit, you will explore a variety of situations that can modelled with an exponential function or its inverse, the logarithmic function. You will learn techniques for solving various problems, such as those posed above.

Looking Ahead

In this unit, you will solve problems involving…

- exponential functions and equations
- logarithmic functions and equations

Unit 3 Project At the Movies

In 2010, Canadian and American movie-goers spent $10.6 billion on tickets, or 33% of the worldwide box office ticket sales. Of the films released in 2010, only 25 were in 3D, but they brought in $2.2 billion of the ticket sales!

You will examine box office revenues for newly released movies, investigate graphs of the revenue over time, determine the function that best represents the data and graph, and use this function to make predictions.

In this project, you will explore the use of mathematics to model box office revenues for a movie of your choice.

CHAPTER 7

Exponential Functions

In the 1920s, watch companies produced glow-in-the-dark dials by using radioluminescent paint, which was made of zinc sulphide mixed with radioactive radium salts. Today, a material called tritium is used in wristwatches and other equipment such as aircraft instruments. In commercial use, the tritium gas is put into tiny vials of borosilicate glass that are placed on the hands and hour markers of a watch dial.

Both radium and tritium are radioactive materials that decay into other elements by emitting different types of radiation. The rate at which radioactive materials decay can be modelled using exponential functions. Exponential functions can also be used to model situations where there is geometric growth, such as in bacterial colonies and financial investments.

In this chapter, you will study exponential functions and use them to solve a variety of problems.

Did You Know?

Radium was once an additive in toothpaste, hair creams, and even food items due to its supposed curative powers. Once it was discovered that radium is over one million times as radioactive as the same mass of uranium, these products were prohibited because of their serious adverse health effects.

Key Terms

exponential function
exponential growth
exponential decay
half-life
exponential equation

Career Link

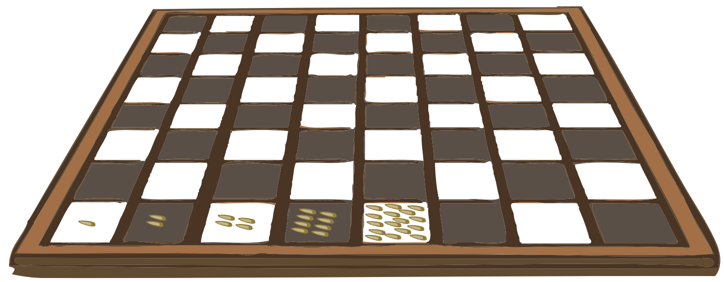
Chemistry helps us understand the world around us and allows us to create new substances and materials. Chemists synthesize, discover, and characterize molecules and their chemical reactions. They may develop products such as synthetic fibres and pharmaceuticals, or processes such as sustainable solutions for energy production and greener methods of manufacturing materials.

Web Link

To learn more about a career in chemistry, go to www.mcgrawhill.ca/school/learningcentres and follow the links.

7.1

Characteristics of Exponential Functions



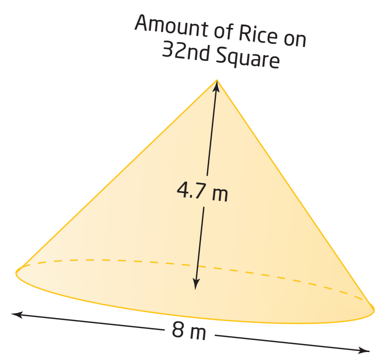
Focus on...

- analysing graphs of exponential functions
- solving problems that involve exponential growth or decay

The following ancient fable from India has several variations, but each makes the same point.

When the creator of the game of chess showed his invention to the ruler of the country, the ruler was so pleased that he gave the inventor the right to name his prize for the invention. The man, who was very wise, asked the king that he be given one grain of rice for the first square of the chessboard, two for the second one, four for the third one, and so on. The ruler quickly accepted the inventor's offer, believing the man had made a mistake in not asking for more.

By the time he was compensated for half the chessboard, the man owned all of the rice in the country, and, by the sixty-fourth square, the ruler owed him almost 20 000 000 000 000 000 000 grains of rice.



The final amount of rice is approximately equal to the volume of a mountain 140 times as tall as Mount Everest. This is an example of how things grow exponentially. What exponential function can be used to model this situation?

Investigate Characteristics of Exponential Functions

Materials

- graphing technology

Explore functions of the form $y = c^x$.

1. Consider the function $y = 2^x$.

a) Graph the function.

b) Describe the shape of the graph.

2. Determine the following and justify your reasoning.

a) the domain and the range of the function $y = 2^x$

b) the y-intercept

c) the x-intercept

d) the equation of the horizontal line (asymptote) that the graph approaches as the values of x get very small

3. Select at least two different values for c in $y = c^x$ that are greater than 2. Graph your functions. Compare each graph to the graph of $y = 2^x$. Describe how the graphs are similar and how they are different.
4. Select at least two different values for c in $y = c^x$ that are between 0 and 1. Graph your functions. How have the graphs changed compared to those from steps 2 and 3?
5. Predict what the graph will look like if $c < 0$. Confirm your prediction using a table of values and a graph.

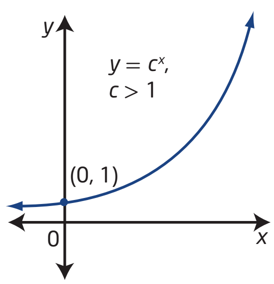
Reflect and Respond

6. **a)** Summarize how the value of c affects the shape and characteristics of the graph of $y = c^x$.

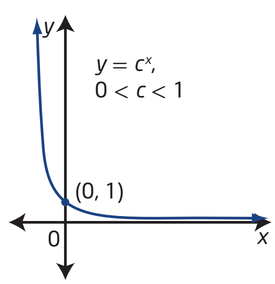
   b) Predict what will happen when $c = 1$. Explain.

Link the Ideas

The graph of an **exponential function**, such as $y = c^x$, is increasing for $c > 1$, decreasing for $0 < c < 1$, and neither increasing nor decreasing for $c = 1$. From the graph, you can determine characteristics such as domain and range, any intercepts, and any asymptotes.



Increasing



Decreasing

exponential function

- a function of the form $y = c^x$, where c is a constant ($c > 0$) and x is a variable

Why is the definition of an exponential function restricted to positive values of c?

Did You Know?

Any letter can be used to represent the base in an exponential function. Some other common forms are $y = a^x$ and $y = b^x$. In this chapter, you will use the letter c. This is to avoid any confusion with the transformation parameters, a, b, h, and k, that you will apply in Section 7.2.

Example 1

Analyse the Graph of an Exponential Function

Graph each exponential function. Then identify the following:

- the domain and range
- the x-intercept and y-intercept, if they exist
- whether the graph represents an increasing or a decreasing function
- the equation of the horizontal asymptote

a) $y = 4^x$

b) $f(x) = \left(\frac{1}{2}\right)^x$

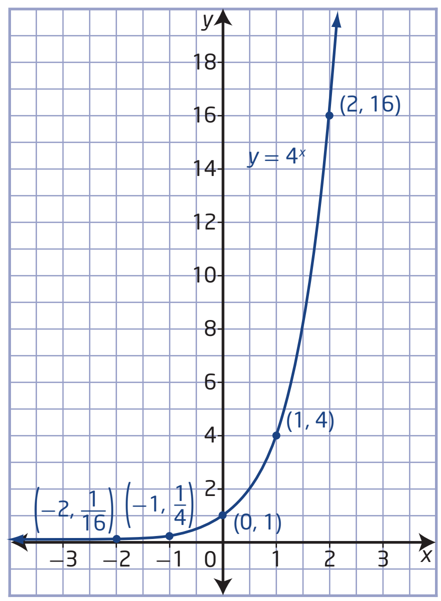
Solution

a) Method 1: Use Paper and Pencil

Use a table of values to graph the function.

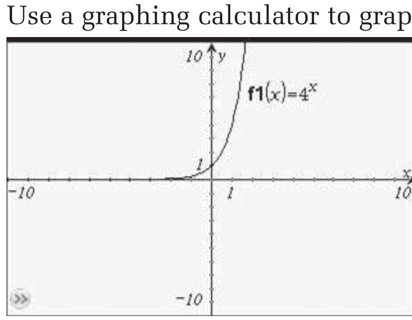
Select integral values of x that make it easy to calculate the corresponding values of y for $y = 4^x$.

x	y
−2	$\frac{1}{16}$
−1	$\frac{1}{4}$
0	1
1	4
2	16



Method 2: Use a Graphing Calculator

Use a graphing calculator to graph $y = 4^x$.



The function is defined for all values of x. Therefore, the domain is $\{x \mid x \in R\}$.

The function has only positive values for y. Therefore, the range is $\{y \mid y > 0, y \in R\}$.

The graph never intersects the x-axis, so there is no x-intercept.

The graph crosses the y-axis at $y = 1$, so the y-intercept is 1.

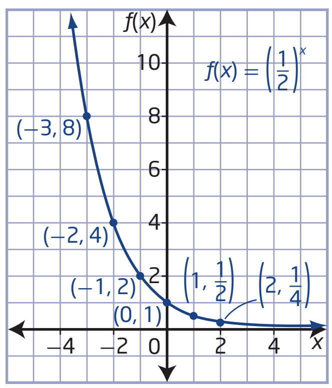
The graph rises to the right throughout its domain, indicating that the values of y increase as the values of x increase. Therefore, the function is increasing over its domain.

Since the graph approaches the line $y = 0$ as the values of x get very small, $y = 0$ is the equation of the horizontal asymptote.

b) **Method 1: Use Paper and Pencil**

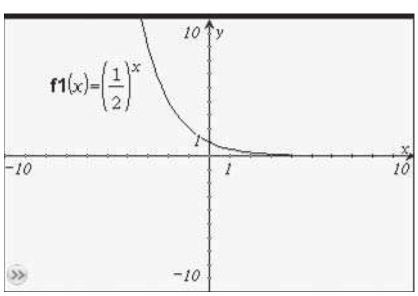
Use a table of values to graph the function.
Select integral values of x that make it easy to calculate the corresponding values of y for $f(x) = \left(\frac{1}{2}\right)^x$.

x	$f(x)$
−3	8
−2	4
−1	2
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$



Method 2: Use a Graphing Calculator

Use a graphing calculator to graph $f(x) = \left(\frac{1}{2}\right)^x$.



The function is defined for all values of x. Therefore, the domain is $\{x \mid x \in \mathrm{R}\}$.

The function has only positive values for y. Therefore, the range is $\{y \mid y > 0, y \in \mathrm{R}\}$.

The graph never intersects the x-axis, so there is no x-intercept.

The graph crosses the y-axis at $y = 1$, so the y-intercept is 1.

Why do the graphs of these exponential functions have a y-intercept of 1?

The graph falls to the right throughout its domain, indicating that the values of y decrease as the values of x increase. Therefore, the function is decreasing over its domain.

Since the graph approaches the line $y = 0$ as the values of x get very large, $y = 0$ is the equation of the horizontal asymptote.

Your Turn

Graph the exponential function $y = 3^x$ without technology. Identify the following:

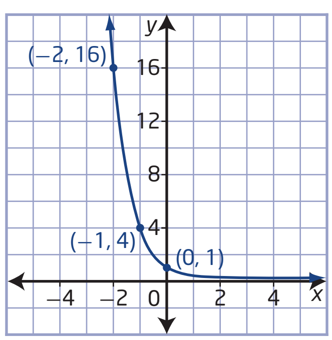
- the domain and range
- the x-intercept and the y-intercept, if they exist
- whether the graph represents an increasing or a decreasing function
- the equation of the horizontal asymptote

Verify your results using graphing technology.

Example 2

Write the Exponential Function Given Its Graph

What function of the form $y = c^x$ can be used to describe the graph shown?



Solution

Look for a pattern in the ordered pairs from the graph.

x	y
−2	16
−1	4
0	1

As the value of x increases by 1 unit, the value of y decreases by a factor of $\frac{1}{4}$. Therefore, for this function, $c = \frac{1}{4}$.

Choose a point other than (0, 1) to substitute into the function $y = \left(\frac{1}{4}\right)^x$ to verify that the function is correct. Try the point (−2, 16).

Why should you not use the point (0, 1) to verify that the function is correct?

Check:

Left Side

y

$= 16$

Right Side

$$\left(\frac{1}{4}\right)^x$$
$$= \left(\frac{1}{4}\right)^{-2}$$
$$= \frac{1}{\left(\frac{1}{4}\right)^2}$$
$$= \left(\frac{4}{1}\right)^2$$
$$= 16$$

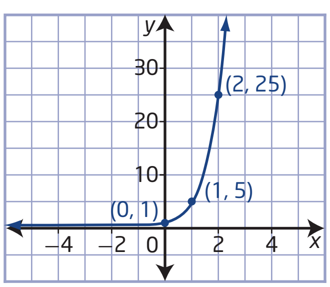
Why is the power with a negative exponent, $\left(\frac{1}{4}\right)^{-2}$, equivalent to the reciprocal of the power with a positive exponent, $\frac{1}{\left(\frac{1}{4}\right)^2}$?

The right side equals the left side, so the function that describes the graph is $y = \left(\frac{1}{4}\right)^x$.

What is another way of expressing this exponential function?

Your Turn

What function of the form $y = c^x$ can be used to describe the graph shown?



Exponential functions of the form $y = c^x$, where $c > 1$, can be used to model **exponential growth**. Exponential functions with $0 < c < 1$ can be used to model **exponential decay**.

exponential growth

- an increasing pattern of values that can be modelled by a function of the form $y = c^x$, where $c > 1$

exponential decay

- a decreasing pattern of values that can be modelled by a function of the form $y = c^x$, where $0 < c < 1$

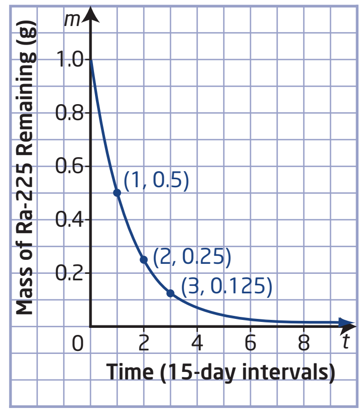
half-life

- the length of time for an unstable element to spontaneously decay to one half its original mass

Example 3

Application of an Exponential Function

A radioactive sample of radium (Ra-225) has a **half-life** of 15 days. The mass, m, in grams, of Ra-225 remaining over time, t, in 15-day intervals, can be modelled using the exponential graph shown.



a) What is the initial mass of Ra-225 in the sample? What value does the mass of Ra-225 remaining approach as time passes?

b) What are the domain and range of this function?

c) Write the exponential decay model that relates the mass of Ra-225 remaining to time, in 15-day intervals.

d) Estimate how many days it would take for Ra-225 to decay to $\frac{1}{30}$ of its original mass.

Solution

a) From the graph, the m-intercept is 1. So, the initial mass of Ra-225 in the sample is 1 g.

The graph is decreasing by a constant factor over time, representing exponential decay. It appears to approach $m = 0$ or 0 g of Ra-225 remaining in the sample.

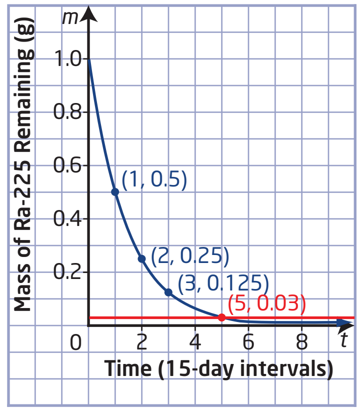
b) From the graph, the domain of the function is $\{t \mid t \geq 0, t \in \mathrm{R}\}$, and the range of the function is $\{m \mid 0 < m \leq 1, m \in \mathrm{R}\}$.

c) The exponential decay model that relates the mass of Ra-225 remaining to time, in 15-day intervals, is the function $m(t) = \left(\frac{1}{2}\right)^t$.

Why is the base of the exponential function $\frac{1}{2}$?

d) Method 1: Use the Graph of the Function

$\frac{1}{30}$ of 1 g is equivalent to 0.0333… or approximately 0.03 g. Locate this approximate value on the vertical axis of the graph and draw a horizontal line until it intersects the graph of the exponential function.



The horizontal line appears to intersect the graph at the point (5, 0.03). Therefore, it takes approximately five 15-day intervals, or 75 days, for Ra-225 to decay to $\frac{1}{30}$ of its original mass.

Method 2: Use a Table of Values

$\frac{1}{30}$ of 1 g is equivalent to $\frac{1}{30}$ g or approximately 0.0333 g.

Create a table of values for $m(t) = \left(\frac{1}{2}\right)^t$.

t	m
1	0.5
2	0.25
3	0.125
4	0.0625
5	0.031 25
6	0.015 625

The table shows that the number of 15-day intervals is between 4 and 5. You can determine a better estimate by looking at values between these numbers.

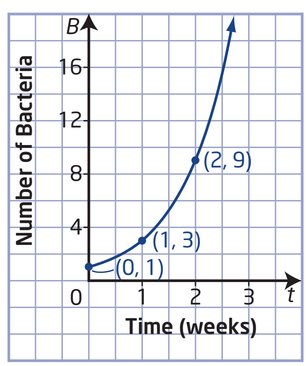
Since 0.0333 is much closer to 0.031 25, try 4.8: $\left(\frac{1}{2}\right)^{4.8} \approx 0.0359$. This is greater than 0.0333.

Try 4.9: $\left(\frac{1}{2}\right)^{4.9} \approx 0.0335$.

Therefore, it will take approximately 4.9 15-day intervals, or 73.5 days, for Ra-225 to decay to $\frac{1}{30}$ of its original mass.

Your Turn

Under ideal circumstances, a certain bacteria population triples every week. This is modelled by the following exponential graph.



a) What are the domain and range of this function?

b) Write the exponential growth model that relates the number, B, of bacteria to the time, t, in weeks.

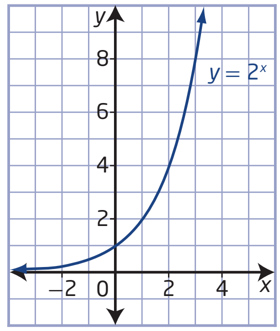
c) Determine approximately how many days it would take for the number of bacteria to increase to eight times the quantity on day 1.

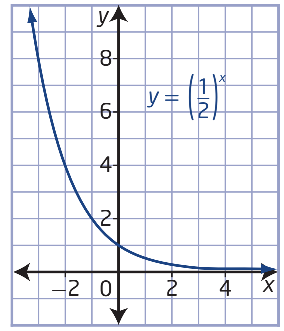
Did You Know?

Exponential functions can be used to model situations involving continuous or discrete data. For example, problems involving radioactive decay are based on continuous data, while those involving populations are based on discrete data. In the case of discrete data, the continuous model will only be valid for a restricted domain.

Key Ideas

- An exponential function of the form $y = c^x$, $c > 0$,
  - is increasing for $c > 1$
  - is decreasing for $0 < c < 1$
  - is neither increasing nor decreasing for $c = 1$
  - has a domain of $\{x \mid x \in R\}$
  - has a range of $\{y \mid y > 0, y \in R\}$
  - has a y-intercept of 1
  - has no x-intercept
  - has a horizontal asymptote at $y = 0$





Check Your Understanding

Practise

1. Decide whether each of the following functions is exponential. Explain how you can tell.

a) $y = x^3$

b) $y = 6^x$

c) $y = x^{\frac{1}{2}}$

d) $y = 0.75^x$

2. Consider the following exponential functions:

- $f(x) = 4^x$
- $g(x) = \left(\frac{1}{4}\right)^x$
- $h(x) = 2^x$

a) Which is greatest when $x = 5$?

b) Which is greatest when $x = -5$?

c) For which value of x do all three functions have the same value? What is this value?

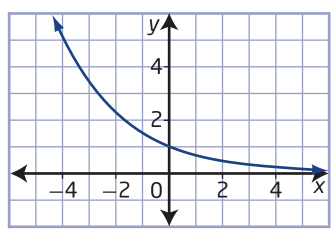
3. Match each exponential function to its corresponding graph.

a) $y = 5^x$

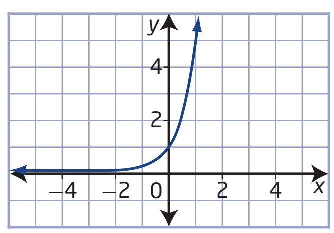
b) $y = \left(\frac{1}{4}\right)^x$

c) $y = \left(\frac{2}{3}\right)^x$

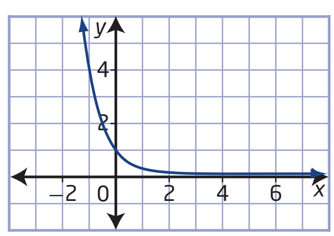
A



B

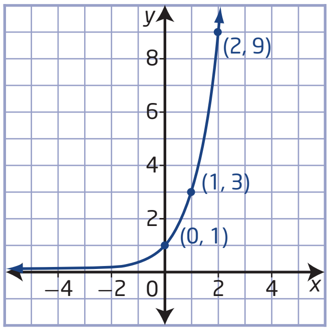


C

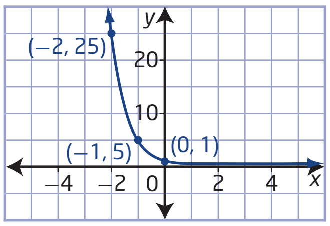


4. Write the function equation for each graph of an exponential function.

a)



b)



5. Sketch the graph of each exponential function. Identify the domain and range, the y-intercept, whether the function is increasing or decreasing, and the equation of the horizontal asymptote.

a) $g(x) = 6^x$

b) $h(x) = 3.2^x$

c) $f(x) = \left(\frac{1}{10}\right)^x$

d) $k(x) = \left(\frac{3}{4}\right)^x$

Apply

6. Each of the following situations can be modelled using an exponential function. Indicate which situations require a value of $c > 1$ (growth) and which require a value of $0 < c < 1$ (decay). Explain your choices.

a) Bacteria in a Petri dish double their number every hour.

b) The half-life of the radioactive isotope actinium-225 is 10 days.

c) As light passes through every 1-m depth of water in a pond, the amount of light available decreases by 20%.

d) The population of an insect colony triples every day.

7. A flu virus is spreading through the student population of a school according to the function $N = 2^t$, where N is the number of people infected and t is the time, in days.

a) Graph the function. Explain why the function is exponential.

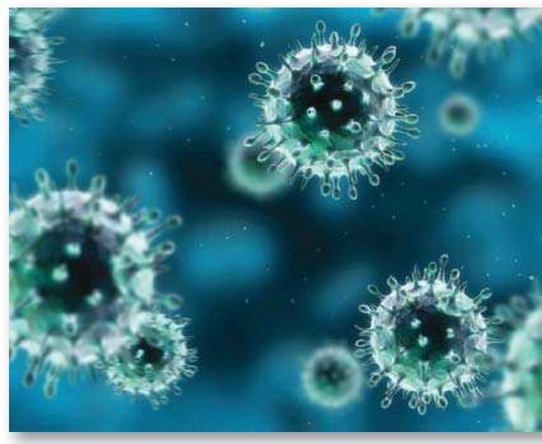
b) How many people have the virus at each time?

i) at the start when $t = 0$

ii) after 1 day

iii) after 4 days

iv) after 10 days



8. If a given population has a constant growth rate over time and is never limited by food or disease, it exhibits exponential growth. In this situation, the growth rate alone controls how quickly (or slowly) the population grows. If a population, P, of fish, in hundreds, experiences exponential growth at a rate of 10% per year, it can be modelled by the exponential function $P(t) = 1.1^t$, where t is time, in years.

a) Why is the base for the exponential function that models this situation 1.1?

b) Graph the function $P(t) = 1.1^t$. What are the domain and range of the function?

c) If the same population of fish decreased at a rate of 5% per year, how would the base of the exponential model change?

d) Graph the new function from part c). What are the domain and range of this function?

9. Scuba divers know that the deeper they dive, the more light is absorbed by the water above them. On a dive, Petra's light meter shows that the amount of light available decreases by 10% for every 10 m that she descends.

a) Write the exponential function that relates the amount, L, as a percent expressed as a decimal, of light available to the depth, d, in 10-m increments.

b) Graph the function.

c) What are the domain and range of the function for this situation?

d) What percent of light will reach Petra if she dives to a depth of 25 m?

10. The CANDU (CANada Deuterium Uranium) reactor is a Canadian-invented pressurized heavy-water reactor that uses uranium-235 (U-235) fuel with a half-life of approximately 700 million years.

a) What exponential function can be used to represent the radioactive decay of 1 kg of U-235? Define the variables you use.

b) Graph the function.

c) How long will it take for 1 kg of U-235 to decay to 0.125 kg?

d) Will the sample in part c) decay to 0 kg? Explain.

Did You Know?

Canada is one of the world's leading uranium producers, accounting for 18% of world primary production. All of the uranium produced in Canada comes from Saskatchewan mines. The energy potential of Saskatchewan's uranium reserves is approximately equivalent to 4.5 billion tonnes of coal or 17.5 billion barrels of oil.

11. Money in a savings account earns compound interest at a rate of 1.75% per year. The amount, A, of money in an account can be modelled by the exponential function $A = P(1.0175)^n$, where P is the amount of money first deposited into the savings account and n is the number of years the money remains in the account.

a) Graph this function using a value of $P = \$1$ as the initial deposit.

b) Approximately how long will it take for the deposit to triple in value?

c) Does the amount of time it takes for a deposit to triple depend on the value of the initial deposit? Explain.

d) In finance, the *rule of 72* is a method of estimating an investment's doubling time when interest is compounded annually. The number 72 is divided by the annual interest rate to obtain the approximate number of years required for doubling. Use your graph and the rule of 72 to approximate the doubling time for this investment.

12. Statistics indicate that the world population since 1995 has been growing at a rate of about 1.27% per year. United Nations records estimate that the world population in 2011 was approximately 7 billion. Assuming the same exponential growth rate, when will the population of the world be 9 billion?

Extend

13. a) On the same set of axes, sketch the graph of the function $y = 5^x$, and then sketch the graph of the inverse of the function by reflecting its graph in the line $y = x$.

b) How do the characteristics of the graph of the inverse of the function relate to the characteristics of the graph of the original exponential function?

c) Express the equation of the inverse of the exponential function in terms of y. That is, write $x = F(y)$.

14. The Krumbein phi scale is used in geology to classify sediments such as silt, sand, and gravel by particle size. The scale is modelled by the function $D(\varphi) = 2^{-\varphi}$, where D is the diameter of the particle, in millimetres, and φ is the Krumbein scale value. Fine sand has a Krumbein scale value of approximately 3. Coarse gravel has a Krumbein scale value of approximately -5.

A sampler showing grains sorted from silt to very coarse sand.

a) Why would a coarse material have a negative scale value?

b) How does the diameter of fine sand compare with the diameter of coarse gravel?

15. Typically, compound interest for a savings account is calculated every month and deposited into the account at that time. The interest could also be calculated daily, or hourly, or even by the second. When the period of time is infinitesimally small, the interest calculation is called continuous compounding. The exponential function that models this situation is $A(t) = Pe^{rt}$, where P is the amount of the initial deposit, r is the annual rate of interest as a decimal value, t is the number of years, and e is the base (approximately equal to 2.7183).

a) Use graphing technology to estimate the doubling period, assuming an annual interest rate of 2% and continuous compounding.

b) Use graphing technology to estimate the doubling period using the compound interest formula $A = P(1 + i)^n$.

c) How do the results in parts a) and b) compare? Which method results in a shorter doubling period?

Did You Know?

The number *e* is irrational because it cannot be expressed as a ratio of integers. It is sometimes called *Euler's number* after the Swiss mathematician Leonhard Euler (pronounced "oiler").

Create Connections

C1 Consider the functions $f(x) = 3x$, $g(x) = x^3$, and $h(x) = 3^x$.

a) Graph each function.

b) List the key features for each function: domain and range, intercepts, and equations of any asymptotes.

c) Identify key features that are common to each function.

d) Identify key features that are different for each function.

C2 Consider the function $f(x) = (-2)^x$.

a) Copy and complete the table of values.

x	$f(x)$
0	
1	
2	
3	
4	
5	

b) Plot the ordered pairs.

c) Do the points form a smooth curve? Explain.

d) Use technology to try to evaluate $f\left(\frac{1}{2}\right)$ and $f\left(\frac{5}{2}\right)$. Use numerical reasoning to explain why these values are undefined.

e) Use these results to explain why exponential functions are defined to only include positive bases.

7.2

Transformations of Exponential Functions

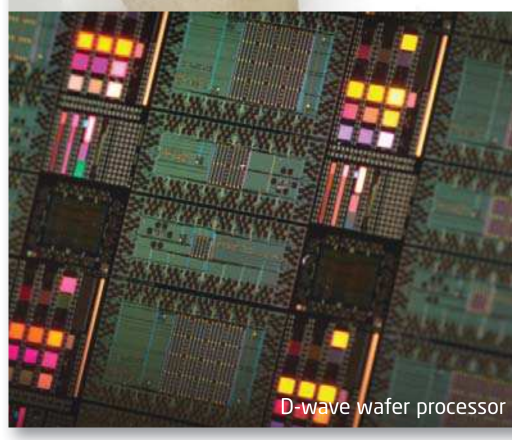
Focus on...

- applying translations, stretches, and reflections to the graphs of exponential functions
- representing these transformations in the equations of exponential functions
- solving problems that involve exponential growth or decay

Transformations of exponential functions are used to model situations such as population growth, carbon dating of samples found at archaeological digs, the physics of nuclear chain reactions, and the processing power of computers.

In this section, you will examine transformations of exponential functions and the impact the transformations have on the corresponding graph.


Taking a sample for carbon dating


D-wave wafer processor

Did You Know?

Moore's law describes a trend in the history of computing hardware. It states that the number of transistors that can be placed on an integrated circuit will double approximately every 2 years. The trend has continued for over half a century.

Investigate Transforming an Exponential Function

Materials

- graphing technology

Apply your prior knowledge of transformations to predict the effects of translations, stretches, and reflections on exponential functions of the form $f(x) = a(c)^{b(x - h)} + k$ and their associated graphs.

A: The Effects of Parameters h and k on the Function $f(x) = a(c)^{b(x - h)} + k$

1. a) Graph each set of functions on one set of coordinate axes. Sketch the graphs in your notebook.

Set A	Set B
i) $f(x) = 3^x$	**i)** $f(x) = 2^x$
ii) $f(x) = 3^x + 2$	**ii)** $f(x) = 2^{x-3}$
iii) $f(x) = 3^x - 4$	**iii)** $f(x) = 2^{x+1}$

b) Compare the graphs in set A. For any constant k, describe the relationship between the graphs of $f(x) = 3^x$ and $f(x) = 3^x + k$.

c) Compare the graphs in set B. For any constant h, describe the relationship between the graphs of $f(x) = 2^x$ and $f(x) = 2^{x-h}$.

Reflect and Respond

2. Describe the roles of the parameters h and k in functions of the form $f(x) = a(c)^{b(x-h)} + k$.

B: The Effects of Parameters *a* and *b* on the Function $f(x) = a(c)^{b(x-h)} + k$

3. a) Graph each set of functions on one set of coordinate axes. Sketch the graphs in your notebook.

Set C	Set D
i) $f(x) = \left(\frac{1}{2}\right)^x$	i) $f(x) = 2^x$
ii) $f(x) = 3\left(\frac{1}{2}\right)^x$	ii) $f(x) = 2^{3x}$
iii) $f(x) = \frac{3}{4}\left(\frac{1}{2}\right)^x$	iii) $f(x) = 2^{\frac{1}{3}x}$
iv) $f(x) = -4\left(\frac{1}{2}\right)^x$	iv) $f(x) = 2^{-2x}$
v) $f(x) = -\frac{1}{3}\left(\frac{1}{2}\right)^x$	v) $f(x) = 2^{-\frac{2}{3}x}$

What effect does the negative sign have on the graph?

b) Compare the graphs in set C. For any real value a, describe the relationship between the graphs of $f(x) = \left(\frac{1}{2}\right)^x$ and $f(x) = a\left(\frac{1}{2}\right)^x$.

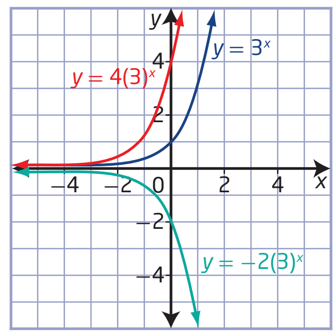
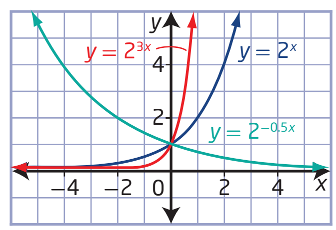
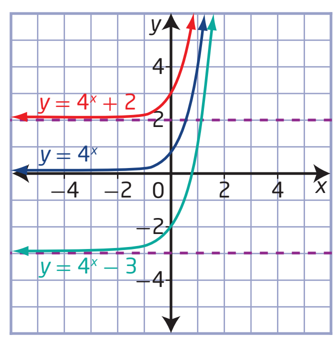
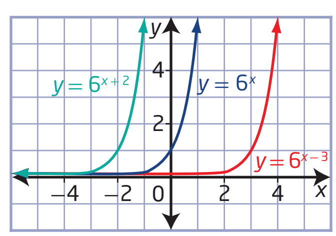
c) Compare the graphs in set D. For any real value b, describe the relationship between the graphs of $f(x) = 2^x$ and $f(x) = 2^{bx}$.

Reflect and Respond

4. Describe the roles of the parameters a and b in functions of the form $f(x) = a(c)^{b(x-h)} + k$.

Link the Ideas

The graph of a function of the form $f(x) = a(c)^{b(x - h)} + k$ is obtained by applying transformations to the graph of the base function $y = c^x$, where $c > 0$.

Parameter	Transformation	Example
a	• Vertical stretch about the x-axis by a factor of $\|a\|$ • For $a < 0$, reflection in the x-axis • $(x, y) \rightarrow (x, ay)$	
b	• Horizontal stretch about the y-axis by a factor of $\frac{1}{\|b\|}$ • For $b < 0$, reflection in the y-axis • $(x, y) \rightarrow \left(\frac{x}{b}, y\right)$	
k	• Vertical translation up or down • $(x, y) \rightarrow (x, y + k)$	
h	• Horizontal translation left or right • $(x, y) \rightarrow (x + h, y)$	

An accurate sketch of a transformed graph is obtained by applying the transformations represented by a and b before the transformations represented by h and k.

How does this compare to your past experience with transformations?

Example 1

Apply Transformations to Sketch a Graph

Consider the base function $y = 3^x$. For each transformed function,

i) state the parameters and describe the corresponding transformations

ii) create a table to show what happens to the given points under each transformation

$y = 3^x$
$\left(-1, \frac{1}{3}\right)$
(0, 1)
(1, 3)
(2, 9)
(3, 27)

iii) sketch the graph of the base function and the transformed function

iv) describe the effects on the domain, range, equation of the horizontal asymptote, and intercepts

a) $y = 2(3)^{x-4}$

b) $y = -\frac{1}{2}(3)^{\frac{1}{5}x} - 5$

Solution

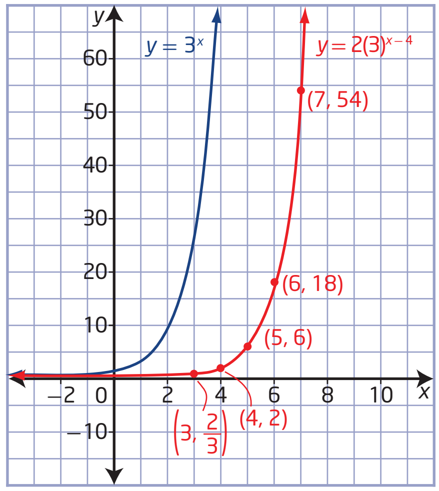
a) **i)** Compare the function $y = 2(3)^{x-4}$ to $y = a(c)^{b(x-h)} + k$ to determine the values of the parameters.

- $b = 1$ corresponds to no horizontal stretch.
- $a = 2$ corresponds to a vertical stretch of factor 2. Multiply the y-coordinates of the points in column 1 by 2.
- $h = 4$ corresponds to a translation of 4 units to the right. Add 4 to the x-coordinates of the points in column 2.
- $k = 0$ corresponds to no vertical translation.

ii) Add columns to the table representing the transformations.

$y = 3^x$	$y = 2(3)^x$	$y = 2(3)^{x-4}$
$\left(-1, \frac{1}{3}\right)$	$\left(-1, \frac{2}{3}\right)$	$\left(3, \frac{2}{3}\right)$
(0, 1)	(0, 2)	(4, 2)
(1, 3)	(1, 6)	(5, 6)
(2, 9)	(2, 18)	(6, 18)
(3, 27)	(3, 54)	(7, 54)

iii) To sketch the graph, plot the points from column 3 and draw a smooth curve through them.



iv) The domain remains the same: $\{x \mid x \in \mathrm{R}\}$.

The range also remains unchanged: $\{y \mid y > 0, y \in \mathrm{R}\}$.

The equation of the asymptote remains as $y = 0$.

There is still no x-intercept, but the y-intercept changes to $\frac{2}{81}$ or approximately 0.025.

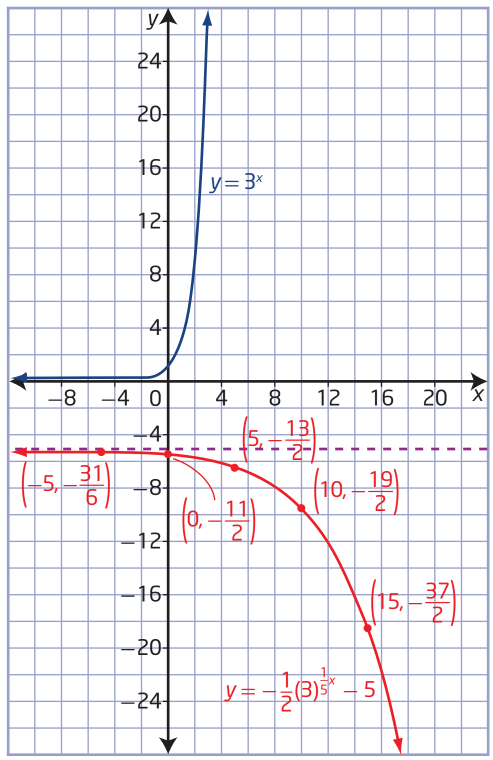
b) **i)** Compare the function $y = -\frac{1}{2}(3)^{\frac{1}{5}x} - 5$ to $y = a(c)^{b(x-h)} + k$ to determine the values of the parameters.

- $b = \frac{1}{5}$ corresponds to a horizontal stretch of factor 5. Multiply the x-coordinates of the points in column 1 by 5.
- $a = -\frac{1}{2}$ corresponds to a vertical stretch of factor $\frac{1}{2}$ and a reflection in the x-axis. Multiply the y-coordinates of the points in column 2 by $-\frac{1}{2}$.
- $h = 0$ corresponds to no horizontal translation.
- $k = -5$ corresponds to a translation of 5 units down. Subtract 5 from the y-coordinates of the points in column 3.

ii) Add columns to the table representing the transformations.

$y = 3^x$	$y = 3^{\frac{1}{5}x}$	$y = -\frac{1}{2}(3)^{\frac{1}{5}x}$	$y = -\frac{1}{2}(3)^{\frac{1}{5}x} - 5$
$\left(-1, \frac{1}{3}\right)$	$\left(-5, \frac{1}{3}\right)$	$\left(-5, -\frac{1}{6}\right)$	$\left(-5, -\frac{31}{6}\right)$
(0, 1)	(0, 1)	$\left(0, -\frac{1}{2}\right)$	$\left(0, -\frac{11}{2}\right)$
(1, 3)	(5, 3)	$\left(5, -\frac{3}{2}\right)$	$\left(5, -\frac{13}{2}\right)$
(2, 9)	(10, 9)	$\left(10, -\frac{9}{2}\right)$	$\left(10, -\frac{19}{2}\right)$
(3, 27)	(15, 27)	$\left(15, -\frac{27}{2}\right)$	$\left(15, -\frac{37}{2}\right)$

iii) To sketch the graph, plot the points from column 4 and draw a smooth curve through them.



Why do the exponential curves have different horizontal asymptotes?

iv) The domain remains the same: $\{x \mid x \in \mathrm{R}\}$.

The range changes to $\{y \mid y < -5, y \in \mathrm{R}\}$ because the graph of the transformed function only exists below the line $y = -5$.

The equation of the asymptote changes to $y = -5$.

There is still no x-intercept, but the y-intercept changes to $-\frac{11}{2}$ or -5.5.

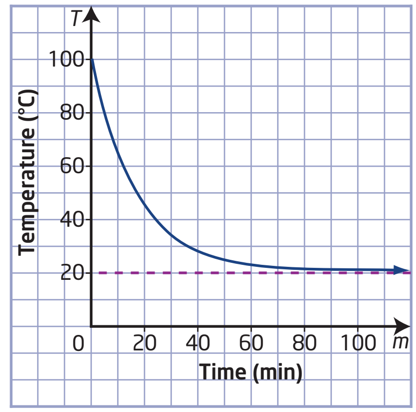
Your Turn

Transform the graph of $y = 4^x$ to sketch the graph of $y = 4^{-2(x+5)} - 3$. Describe the effects on the domain, range, equation of the horizontal asymptote, and intercepts.

Example 2

Use Transformations of an Exponential Function to Model a Situation

A cup of water is heated to 100 °C and then allowed to cool in a room with an air temperature of 20 °C. The temperature, T, in degrees Celsius, is measured every minute as a function of time, m, in minutes, and these points are plotted on a coordinate grid. It is found that the temperature of the water decreases exponentially at a rate of 25% every 5 min. A smooth curve is drawn through the points, resulting in the graph shown.



a) What is the transformed exponential function in the form $y = a(c)^{b(x-h)} + k$ that can be used to represent this situation?

b) Describe how each of the parameters in the transformed function relates to the information provided.

Solution

a) Since the water temperature decreases by 25% for each 5-min time interval, the base function must be $T(t) = \left(\frac{3}{4}\right)^t$, where T is the temperature and t is the time, in 5-min intervals.

Why is the base of the exponential function $\frac{3}{4}$ when the temperature is reduced by 25%?

The exponent t can be replaced by the rational exponent $\frac{m}{5}$, where m represents the number of minutes: $T(m) = \left(\frac{3}{4}\right)^{\frac{m}{5}}$.

What is the value of the exponent $\frac{m}{5}$ when $m = 5$? How does this relate to the exponent t in the first version of the function?

The asymptote at $T = 20$ means that the function has been translated vertically upward 20 units. This is represented in the function as

$T(m) = \left(\frac{3}{4}\right)^{\frac{m}{5}} + 20.$

The T-intercept of the graph occurs at (0, 100). So, there must be a vertical stretch factor, a. Use the coordinates of the T-intercept to determine a.

$$\begin{aligned} T(m) &= a\left(\frac{3}{4}\right)^{\frac{m}{5}} + 20 \\ 100 &= a\left(\frac{3}{4}\right)^{\frac{0}{5}} + 20 \\ 100 &= a(1) + 20 \\ 80 &= a \end{aligned}$$

Substitute $a = 80$ into the function: $T(m) = 80\left(\frac{3}{4}\right)^{\frac{m}{5}} + 20$.

Check: Substitute $m = 20$ into the function. Compare the result to the graph.

$$T(m) = 80\left(\frac{3}{4}\right)^{\frac{m}{5}} + 20$$
$$T(20) = 80\left(\frac{3}{4}\right)^{\frac{20}{5}} + 20$$
$$= 80\left(\frac{3}{4}\right)^{4} + 20$$
$$= 80\left(\frac{81}{256}\right) + 20$$
$$= 45.3125$$

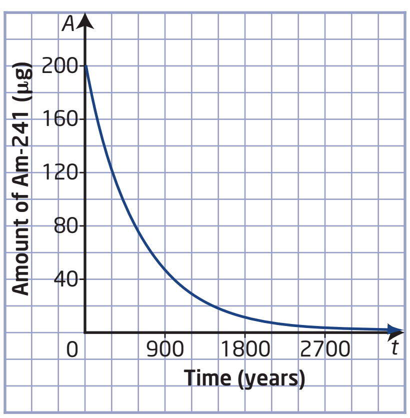
From the graph, the value of T when $m = 20$ is approximately 45. This matches the calculated value. Therefore, the transformed function that models the water temperature as it cools is $T(m) = 80\left(\frac{3}{4}\right)^{\frac{m}{5}} + 20$.

b) Based on the function $y = a(c)^{b(x-h)} + k$, the parameters of the transformed function are

- $b = \frac{1}{5}$, representing the interval of time, 5 min, over which a 25% decrease in temperature of the water occurs
- $a = 80$, representing the difference between the initial temperature of the heated cup of water and the air temperature of the room
- $h = 0$, representing the start time of the cooling process
- $k = 20$, representing the air temperature of the room

Your Turn

The radioactive element americium (Am) is used in household smoke detectors. Am-241 has a half-life of approximately 432 years. The average smoke detector contains 200 μg of Am-241.



a) What is the transformed exponential function that models the graph showing the radioactive decay of 200 μg of Am-241?

b) Identify how each of the parameters of the function relates to the transformed graph.

Did You Know?

In SI units, the symbol "μg" represents a microgram, or one millionth of a gram. In the medical field, the symbol "mcg" is used to avoid any confusion with milligrams (mg) in written prescriptions.

Key Ideas

- To sketch the graph of an exponential function of the form $y = a(c)^{b(x-h)} + k$, apply transformations to the graph of $y = c^x$, where $c > 0$. The transformations represented by a and b may be applied in any order before the transformations represented by h and k.
- The parameters a, b, h, and k in exponential functions of the form $y = a(c)^{b(x-h)} + k$ correspond to the following transformations:
  - a corresponds to a vertical stretch about the x-axis by a factor of $|a|$ and, if $a < 0$, a reflection in the x-axis.
  - b corresponds to a horizontal stretch about the y-axis by a factor of $\frac{1}{|b|}$ and, if $b < 0$, a reflection in the y-axis.
  - h corresponds to a horizontal translation left or right.
  - k corresponds to a vertical translation up or down.
- Transformed exponential functions can be used to model real-world applications of exponential growth or decay.

Check Your Understanding

Practise

1. Match each function with the corresponding transformation of $y = 3^x$.

a) $y = 2(3)^x$ **b)** $y = 3^{x-2}$

c) $y = 3^x + 4$ **d)** $y = 3^{\frac{x}{5}}$

A translation up

B horizontal stretch

C vertical stretch

D translation right

2. Match each function with the corresponding transformation of $y = \left(\frac{3}{5}\right)^x$.

a) $y = \left(\frac{3}{5}\right)^{x+1}$ **b)** $y = -\left(\frac{3}{5}\right)^x$

c) $y = \left(\frac{3}{5}\right)^{-x}$ **d)** $y = \left(\frac{3}{5}\right)^x - 2$

A reflection in the x-axis

B reflection in the y-axis

C translation down

D translation left

3. For each function, identify the parameters a, b, h, and k and the type of transformation that corresponds to each parameter.

a) $f(x) = 2(3)^x - 4$

b) $g(x) = 6^{x-2} + 3$

c) $m(x) = -4(3)^{x+5}$

d) $y = \left(\frac{1}{2}\right)^{3(x-1)}$

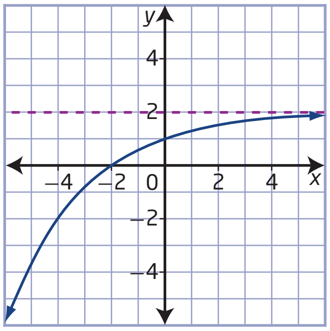
e) $n(x) = -\frac{1}{2}(5)^{2(x-4)} + 3$

f) $y = -\left(\frac{2}{3}\right)^{2x-2}$

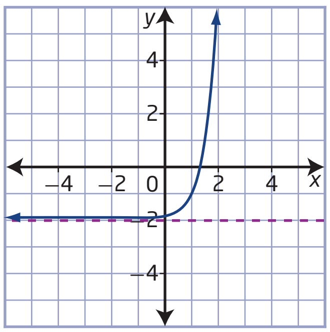
g) $y = 1.5(0.75)^{\frac{x-4}{2}} - \frac{5}{2}$

4. Without using technology, match each graph with the corresponding function. Justify your choice.

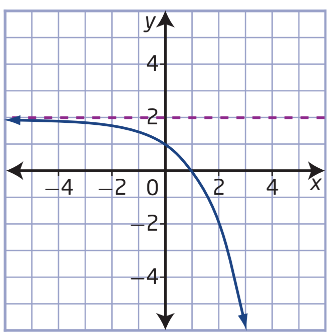
a)



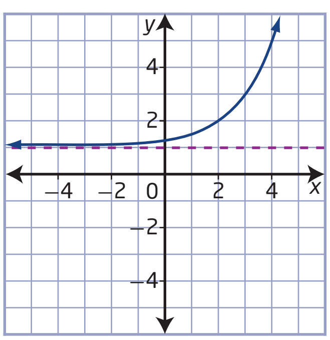
b)



c)



d)



A $y = 3^{2(x-1)} - 2$

B $y = 2^{x-2} + 1$

C $y = -\left(\frac{1}{2}\right)^{\frac{1}{2}x} + 2$

D $y = -\frac{1}{2}(4)^{\frac{1}{2}(x+1)} + 2$

5. The graph of $y = 4^x$ is transformed to obtain the graph of $y = \frac{1}{2}(4)^{-(x-3)} + 2$.

a) What are the parameters and corresponding transformations?

b) Copy and complete the table.

$y = 4^x$	$y = 4^{-x}$	$y = \frac{1}{2}(4)^{-x}$	$y = \frac{1}{2}(4)^{-(x-3)} + 2$
$\left(-2, \frac{1}{16}\right)$			
$\left(-1, \frac{1}{4}\right)$			
(0, 1)			
(1, 4)			
(2, 16)			

c) Sketch the graph of $y = \frac{1}{2}(4)^{-(x-3)} + 2$.

d) Identify the domain, range, equation of the horizontal asymptote, and any intercepts for the function $y = \frac{1}{2}(4)^{-(x-3)} + 2$.

6. For each function,

i) state the parameters a, b, h, and k

ii) describe the transformation that corresponds to each parameter

iii) sketch the graph of the function

iv) identify the domain, range, equation of the horizontal asymptote, and any intercepts

a) $y = 2(3)^x + 4$

b) $m(r) = -(2)^{r-3} + 2$

c) $y = \frac{1}{3}(4)^{x+1} + 1$

d) $n(s) = -\frac{1}{2}\left(\frac{1}{3}\right)^{\frac{1}{4}s} - 3$

Apply

7. Describe the transformations that must be applied to the graph of each exponential function $f(x)$ to obtain the transformed function. Write each transformed function in the form $y = a(c)^{b(x-h)} + k$.

a) $f(x) = \left(\frac{1}{2}\right)^x$, $y = f(x - 2) + 1$

b) $f(x) = 5^x$, $y = -0.5f(x - 3)$

c) $f(x) = \left(\frac{1}{4}\right)^x$, $y = -f(3x) + 1$

d) $f(x) = 4^x$, $y = 2f\left(-\frac{1}{3}(x - 1)\right) - 5$

8. For each pair of exponential functions in #7, sketch the original and transformed functions on the same set of coordinate axes. Explain your procedure.

9. The persistence of drugs in the human body can be modelled using an exponential function. Suppose a new drug follows the model $M(h) = M_0(0.79)^{\frac{h}{3}}$, where M is the mass, in milligrams, of drug remaining in the body; M_0 is the mass, in milligrams, of the dose taken; and h is the time, in hours, since the dose was taken.

   a) Explain the roles of the numbers 0.79 and $\frac{1}{3}$.

   b) A standard dose is 100 mg. Sketch the graph showing the mass of the drug remaining in the body for the first 48 h.

   c) What does the M-intercept represent in this situation?

   d) What are the domain and range of this function?

10. The rate at which liquids cool can be modelled by an approximation of Newton's law of cooling, $T(t) = (T_i - T_f)(0.9)^{\frac{t}{5}} + T_f$, where T_f represents the final temperature, in degrees Celsius; T_i represents the initial temperature, in degrees Celsius; and t represents the elapsed time, in minutes. Suppose a cup of coffee is at an initial temperature of 95 °C and cools to a temperature of 20 °C.

    a) State the parameters a, b, h, and k for this situation. Describe the transformation that corresponds to each parameter.

    b) Sketch a graph showing the temperature of the coffee over a period of 200 min.

    c) What is the approximate temperature of the coffee after 100 min?

    d) What does the horizontal asymptote of the graph represent?

11. A biologist places agar, a gel made from seaweed, in a Petri dish and infects it with bacteria. She uses the measurement of the growth ring to estimate the number of bacteria present. The biologist finds that the bacteria increase in population at an exponential rate of 20% every 2 days.

    a) If the culture starts with a population of 5000 bacteria, what is the transformed exponential function in the form $P = a(c)^{bx}$ that represents the population, P, of the bacteria over time, x, in days?

    b) Describe the parameters used to create the transformed exponential function.

    c) Graph the transformed function and use it to predict the bacteria population after 9 days.

12. Living organisms contain carbon-12 (C-12), which does not decay, and carbon-14 (C-14), which does. When an organism dies, the amount of C-14 in its tissues decreases exponentially with a half-life of about 5730 years.

    a) What is the transformed exponential function that represents the percent, P, of C-14 remaining after t years?

    b) Graph the function and use it to determine the approximate age of a dead organism that has 20% of the original C-14 present in its tissues.

Did You Know?

Carbon dating can only be used to date organic material, or material from once-living things. It is only effective in dating organisms that lived up to about 60 000 years ago.

Extend

13. On Monday morning, Julia found that a colony of bacteria covered an area of 100 cm^2 on the agar. After 10 h, she found that the area had increased to 200 cm^2. Assume that the growth is exponential.

a) By Tuesday morning (24 h later), what area do the bacteria cover?

b) Consider Earth to be a sphere with radius 6378 km. How long would these bacteria take to cover the surface of Earth?

14. Fifteen years ago, the fox population of a national park was 325 foxes. Today, it is 650 foxes. Assume that the population has experienced exponential growth.

a) Project the fox population in 20 years.

b) What is one factor that might slow the growth rate to less than exponential? Is exponential growth healthy for a population? Why or why not?

Create Connections

C1 The graph of an exponential function of the form $y = c^x$ does not have an x-intercept. Explain why this occurs using an example of your own.

C2 a) Which parameters of an exponential function affect the x-intercept of the graph of the function? Explain.

b) Which parameters of an exponential function affect the y-intercept of the graph of the function? Explain.

Project Corner — Modelling a Curve

It is not easy to determine the best mathematical model for real data. In many situations, one model works best for a limited period of time, and then another model is better. Work with a partner. Let x represent the time, in weeks, and let y represent the cumulative box office revenue, in millions of dollars.

- The curves for *Avatar* and *Dark Knight* appear to have a horizontal asymptote. What do you think this represents in this context? Do you think the curve for *Titanic* will eventually exhibit this characteristic as well? Explain.
- Consider the curve for *Titanic*.
  - If the vertex is located at (22, 573), determine a quadratic function of the form $y = a(x - h)^2 + k$ that might model this portion of the curve.
  - Suppose that the curve has a horizontal asymptote with equation $y = 600$. Determine an exponential function of the form $y = -35(0.65)^{0.3(x-h)} + k$ that might model the curve.
  - Which type of function do you think better models this curve? Explain.

7.3

Solving Exponential Equations

Focus on...

- determining the solution of an exponential equation in which the bases are powers of one another
- solving problems that involve exponential growth or decay
- solving problems that involve the application of exponential equations to loans, mortgages, and investments

Banks, credit unions, and investment firms often have financial calculators on their Web sites. These calculators use a variety of formulas, some based on exponential functions, to help users calculate amounts such as annuity values or compound interest in savings accounts. For compound interest calculators, users must input the dollar amount of the initial deposit; the amount of time the money is deposited, called the term; and the interest rate the financial institution offers.

Did You Know?

In 1935, the first Franco-Albertan credit union was established in Calgary.

Investigate the Different Ways to Express Exponential Expressions

Materials

- graphing technology

1. a) Copy the table into your notebook and complete it by

- substituting the value of n into each exponential expression
- using your knowledge of exponent laws to rewrite each expression as an equivalent expression with base 2

n	$\left(\frac{1}{2}\right)^n$	2^n	4^n
-2	$\left(\frac{1}{2}\right)^{-2} = (2^{-1})^{-2}$ $= 2^2$		
-1			
0			
1			
2			$4^2 = (2^2)^2$ $= 2^4$

b) What patterns do you observe in the equivalent expressions? Discuss your findings with a partner.

2. For each exponential expression in the column for 2^n, identify the equivalent exponential expressions with different bases in the other expression columns.

Reflect and Respond

3. a) Explain how to rewrite the **exponential equation** $2^x = 8^{x-1}$ so that the bases are the same.

b) Describe how you could use this information to solve for x. Then, solve for x.

c) Graph the exponential functions on both sides of the equation in part a) on the same set of axes. Explain how the point of intersection of the two graphs relates to the solution you determined in part b).

4. a) Consider the exponential equation $3^x = 4^{2x-1}$. Can this equation be solved in the same way as the one in step 3a)? Explain.

b) What are the limitations when solving exponential equations that have terms with different bases?

exponential equation

- an equation that has a variable in an exponent

Link the Ideas

Exponential expressions can be written in different ways. It is often useful to rewrite an exponential expression using a different base than the one that is given. This is helpful when solving exponential equations because the exponents of exponential expressions with the same base can be equated to each other. For example,

$$4^{2x} = 8^{x+1}$$

$$(2^2)^{2x} = (2^3)^{x+1} \qquad \text{Express the base on each side as a power of 2.}$$

$$2^{4x} = 2^{3x+3}$$

Since the bases on both sides of the equation are now the same, the exponents must be equal.

Is this statement true for all bases? Explain.

$$4x = 3x + 3 \qquad \text{Equate exponents.}$$

$$x = 3$$

This method of solving an exponential equation is based on the property that if $c^x = c^y$, then $x = y$, for $c \neq -1, 0, 1$.

Example 1

Change the Base of Powers

Rewrite each expression as a power with a base of 3.

a) 27 **b)** 9^2 **c)** $27^{\frac{1}{3}}\left(\sqrt[3]{81}\right)^2$

Solution

a) $27 = 3^3$ 27 is the third power of 3.

b) $9^2 = (3^2)^2$ Write 9 as 3^2.

$= 3^4$ Apply the power of a power law.

c)
$$
\begin{aligned}
27^{\frac{1}{3}}\left(\sqrt[3]{81}\right)^2 &= 27^{\frac{1}{3}}\left(81^{\frac{2}{3}}\right) && \text{Write the radical in exponential form.}\\
&= (3^3)^{\frac{1}{3}}(3^4)^{\frac{2}{3}} && \text{Express the bases as powers of 3.}\\
&= 3^1\left(3^{\frac{8}{3}}\right) && \text{Apply the power of a power law.}\\
&= 3^{1+\frac{8}{3}} && \text{Apply the product of powers law.}\\
&= 3^{\frac{11}{3}} && \text{Simplify.}
\end{aligned}
$$

Your Turn

Write each expression as a power with base 2.

a) 4^3 **b)** $\dfrac{1}{8}$ **c)** $8^{\frac{2}{3}}\left(\sqrt{16}\right)^3$

Example 2

Solve an Equation by Changing the Base

Solve each equation.

a) $4^{x+2} = 64^x$ **b)** $4^{2x} = 8^{2x-3}$

Solution

a) Method 1: Apply a Change of Base

$4^{x+2} = 64^x$

$4^{x+2} = (4^3)^x$ Express the base on the right side as a power with base 4.

$4^{x+2} = 4^{3x}$ Apply the power of a power law.

Since both sides are single powers of the same base, the exponents must be equal.

Equate the exponents.

$x + 2 = 3x$

$2 = 2x$ Isolate the term containing x.

$x = 1$

Check:

Left Side	Right Side
4^{x+2}	64^x
$= 4^{1+2}$	$= 64^1$
$= 4^3$	$= 64$
$= 64$	

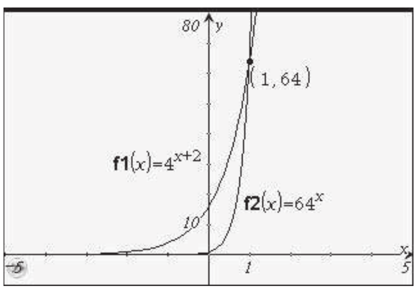
Left Side = Right Side

The solution is $x = 1$.

Method 2: Use a Graphing Calculator

Enter the left side of the equation as one function and the right side as another function. Identify where the graphs intersect using the intersection feature.

You may have to adjust the window settings to view the point of intersection.



The graphs intersect at the point (1, 64).

The solution is $x = 1$.

b)

$$\begin{aligned} 4^{2x} &= 8^{2x-3} \\ (2^2)^{2x} &= (2^3)^{2x-3} && \text{Express the bases on both sides as powers of 2.} \\ 2^{4x} &= 2^{6x-9} && \text{Apply the power of a power law.} \\ 4x &= 6x - 9 && \text{Equate the exponents.} \\ -2x &= -9 && \text{Isolate the term containing } x. \\ x &= \frac{9}{2} && \text{Solve for } x. \end{aligned}$$

Check:

Left Side	Right Side
4^{2x}	8^{2x-3}
$= 4^{2\left(\frac{9}{2}\right)}$	$= 8^{2\left(\frac{9}{2}\right)-3}$
$= 4^9$	$= 8^{9-3}$
$= 262\ 144$	$= 8^6$
	$= 262\ 144$

Left Side = Right Side

The solution is $x = \frac{9}{2}$.

Your Turn

Solve. Check your answers using graphing technology.

a) $2^{4x} = 4^{x+3}$ **b)** $9^{4x} = 27^{x-1}$

Example 3

Solve Problems Involving Exponential Equations With Different Bases

Christina plans to buy a car. She has saved \$5000. The car she wants costs \$5900. How long will Christina have to invest her money in a term deposit that pays 6.12% per year, compounded quarterly, before she has enough to buy the car?

Solution

The formula for compound interest is $A = P(1 + i)^n$, where A is the amount of money at the end of the investment; P is the principal amount deposited; i is the interest rate per compounding period, expressed as a decimal; and n is the number of compounding periods.

In this problem:

$A = 5900$

$P = 5000$

$i = 0.0612 \div 4$ or 0.0153

Divide the interest rate by 4 because interest is paid quarterly or four times a year.

Substitute the known values into the formula.

$$A = P(1 + r)^n$$
$$5900 = 5000(1 + 0.0153)^n$$
$$1.18 = 1.0153^n$$

You will learn how to solve equations like this algebraically when you study logarithms in Chapter 8.

The exponential equation consists of bases that cannot be changed into the same form without using more advanced mathematics.

Method 1: Use Systematic Trial

Use systematic trial to find the approximate value of n that satisfies this equation.

Substitute an initial guess into the equation and evaluate the result. Adjust the estimated solution according to whether the result is too high or too low.

Try $n = 10$.

Why choose whole numbers for n?

$1.0153^{10} = 1.1639\ldots$, which is less than 1.18.

The result is less than the left side of the equation, so try a value of $n = 14$.

$1.0153^{14} = 1.2368\ldots$, which is greater than 1.18.

The result is more than the left side of the equation, so try a value of $n = 11$.

$1.0153^{11} = 1.1817\ldots$, which is approximately equal to 1.18.

The number of compounding periods is approximately 11.

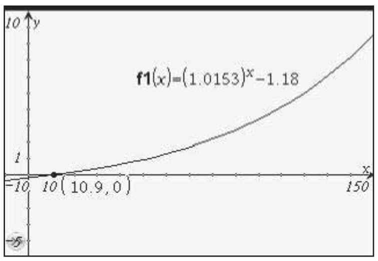
Since interest is paid quarterly, there are four compounding periods in each year. Therefore, it will take approximately $\frac{11}{4}$ or 2.75 years for Christina's investment to reach a value of \$5900.

Method 2: Use a Graphing Calculator

Enter the single function $y = 1.0153^x - 1.18$ and identify where the graph intersects the x-axis.

How is this similar to graphing the left side and right side of the equation and determining where the two graphs intersect?

You may have to adjust the window settings to view the point of intersection.



Use the features of the graphing calculator to show that the zero of the function is approximately 11.

Since interest is paid quarterly, there are four compounding periods in each year. Therefore, it will take approximately $\frac{11}{4}$ or 2.75 years for Christina's investment to reach a value of $5900.

Your Turn

Determine how long $1000 needs to be invested in an account that earns 8.3% compounded semi-annually before it increases in value to $1490.

Key Ideas

- Some exponential equations can be solved directly if the terms on either side of the equal sign have the same base or can be rewritten so that they have the same base.
  - If the bases are the same, then equate the exponents and solve for the variable.
  - If the bases are different but can be rewritten with the same base, use the exponent laws, and then equate the exponents and solve for the variable.
- Exponential equations that have terms with bases that you cannot rewrite using a common base can be solved approximately. You can use either of the following methods:
  - Use systematic trial. First substitute a reasonable estimate for the solution into the equation, evaluate the result, and adjust the next estimate according to whether the result is too high or too low. Repeat this process until the sides of the equation are approximately equal.
  - Graph the functions that correspond to the expressions on each side of the equal sign, and then identify the value of x at the point of intersection, or graph as a single function and find the x-intercept.

Check Your Understanding

Practise

1. Write each expression with base 2.

a) 4^6 **b)** 8^3

c) $\left(\frac{1}{8}\right)^2$ **d)** 16

2. Rewrite the expressions in each pair so that they have the same base.

a) 2^3 and 4^2

b) 9^x and 27

c) $\left(\frac{1}{2}\right)^{2x}$ and $\left(\frac{1}{4}\right)^{x-1}$

d) $\left(\frac{1}{8}\right)^{x-2}$ and 16^x

3. Write each expression as a single power of 4.

a) $(\sqrt{16})^2$ **b)** $\sqrt[3]{16}$

c) $\sqrt{16}\,(\sqrt[3]{64})^2$ **d)** $(\sqrt{2})^8(\sqrt[4]{4})^4$

4. Solve. Check your answers using substitution.

a) $2^{4x} = 4^{x+3}$ **b)** $25^{x-1} = 5^{3x}$

c) $3^{w+1} = 9^{w-1}$ **d)** $36^{3m-1} = 6^{2m+5}$

5. Solve. Check your answers using graphing technology.

a) $4^{3x} = 8^{x-3}$ **b)** $27^x = 9^{x-2}$

c) $125^{2y-1} = 25^{y+4}$ **d)** $16^{2k-3} = 32^{k+3}$

6. Solve for x using systematic trial. Check your answers using graphing technology. Round answers to one decimal place.

a) $2 = 1.07^x$ **b)** $3 = 1.1^x$

c) $0.5 = 1.2^{x-1}$ **d)** $5 = 1.08^{x+2}$

7. Solve for t graphically. Round answers to two decimal places, if necessary.

a) $100 = 10(1.04)^t$ **b)** $10 = \left(\frac{1}{2}\right)^{2t}$

c) $12 = \left(\frac{1}{4}\right)^{\frac{t}{3}}$ **d)** $100 = 25\left(\frac{1}{2}\right)^{\frac{t}{4}}$

e) $2^t = 3^{t-1}$ **f)** $5^{t-2} = 4^t$

g) $8^{t+1} = 3^{t-1}$ **h)** $7^{2t+1} = 4^{t-2}$

Apply

8. If seafood is not kept frozen (below 0 °C), it will spoil due to bacterial growth. The relative rate of spoilage increases with temperature according to the model $R = 100(2.7)^{\frac{T}{8}}$, where T is the temperature, in degrees Celsius, and R is the relative spoilage rate.

a) Sketch a graph of the relative spoilage rate R versus the temperature T from 0 °C to 25 °C.

b) Use your graph to predict the temperature at which the relative spoilage rate doubles to 200.

c) What is the relative spoilage rate at 15 °C?

d) If the maximum acceptable relative spoilage rate is 500, what is the maximum storage temperature?

Did You Know?

The relative rate of spoilage for seafood is defined as the shelf life at 0° C divided by the shelf life at temperature T, in degrees Celsius.

9. A bacterial culture starts with 2000 bacteria and doubles every 0.75 h. After how many hours will the bacteria count be 32 000?

10. Simionie needs \$7000 to buy a snowmobile, but only has \$6000. His bank offers a GIC that pays an annual interest rate of 3.93%, compounded annually. How long would Simionie have to invest his money in the GIC to have enough money to buy the snowmobile?

Did You Know?

A Guaranteed Investment Certificate (GIC) is a secure investment that guarantees 100% of the original amount that is invested. The investment earns interest, at either a fixed or a variable rate, based on a predetermined formula.

11. A $1000 investment earns interest at a rate of 8% per year, compounded quarterly.
   a) Write an equation for the value of the investment as a function of time, in years.
   b) Determine the value of the investment after 4 years.
   c) How long will it take for the investment to double in value?

12. Cobalt-60 (Co-60) has a half-life of approximately 5.3 years.
   a) Write an exponential function to model this situation.
   b) What fraction of a sample of Co-60 will remain after 26.5 years?
   c) How long will it take for a sample of Co-60 to decay to $\frac{1}{512}$ of its original mass?

13. A savings bond offers interest at a rate of 6.6% per year, compounded semi-annually. Suppose that you buy a $500 bond.
   a) Write an equation for the value of the investment as a function of time, in years.
   b) Determine the value of the investment after 5 years.
   c) How long will it take for the bond to triple in value?

14. Glenn and Arlene plan to invest money for their newborn grandson so that he has $20 000 available for his education on his 18th birthday. Assuming a growth rate of 7% per year, compounded semi-annually, how much will Glenn and Arlene need to invest today?

Did You Know?

When the principal, P, needed to generate a future amount is unknown, you can rearrange the compound interest formula to isolate P: $P = A(1 + i)^{-n}$. In this form, the principal is referred to as the present value and the amount is referred to as the future value. Then, you can calculate the present value, PV, the amount that must be invested or borrowed today to result in a specific future value, FV, using the formula $PV = FV(1 + i)^{-n}$, where i is the interest rate per compounding period, expressed as a decimal value, and n is the number of compounding periods.

Extend

15. a) Solve each inequality.
   i) $2^{3x} > 4^{x+1}$ ii) $81^x < 27^{2x+1}$
   b) Use a sketch to help you explain how you can use graphing technology to check your answers.
   c) Create an inequality involving an exponential expression. Solve the inequality graphically.

16. Does the equation $4^{2x} + 2(4^x) - 3 = 0$ have any real solutions? Explain your answer.

17. If $4^x - 4^{x-1} = 24$, what is the value of $(2^x)^x$?

18. The formula for calculating the monthly mortgage payment, PMT, for a property is $PMT = PV\left[\frac{i}{1 - (1 + i)^{-n}}\right]$, where PV is the present value of the mortgage; i is the interest rate per compounding period, as a decimal; and n is the number of payment periods. To buy a house, Tyseer takes out a mortgage worth $150 000 at an equivalent monthly interest rate of 0.25%. He can afford monthly mortgage payments of $831.90. Assuming the interest rate and monthly payments stay the same, how long will it take Tyseer to pay off the mortgage?

Create Connections

C1 a) Explain how you can write 16^2 with base 4.
   b) Explain how you can write 16^2 with two other, different, bases.

C2 The steps for solving the equation $16^{2x} = 8^{x-3}$ are shown below, but in a jumbled order.

$$2^{8x} = 2^{3x-9}$$
$$16^{2x} = 8^{x-3}$$
$$x = -\frac{9}{5}$$
$$8x = 3x - 9$$
$$(2^4)^{2x} = (2^3)^{x-3}$$
$$5x = -9$$

   a) Copy the steps into your notebook, rearranged in the correct order.
   b) Write a brief explanation beside each step.

Chapter 7 Review

7.1 Characteristics of Exponential Functions, pages 334–345

1. Match each item in set A with its graph from set B.

Set A

a) The population of a country, in millions, grows at a rate of 1.5% per year.

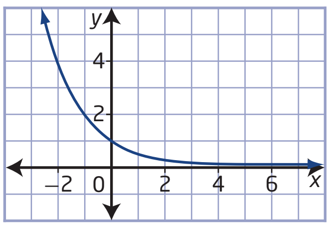
b) $y = 10^x$

c) Tungsten-187 is a radioactive isotope that has a half-life of 1 day.

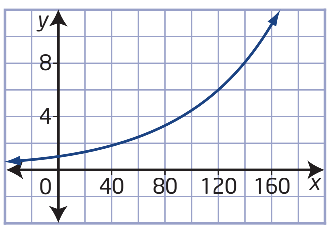
d) $y = 0.2^x$

Set B

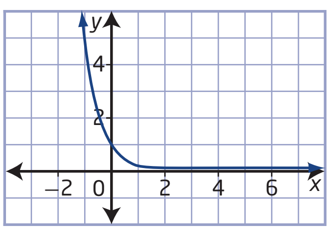
A



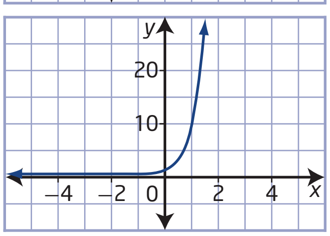
B



C



D

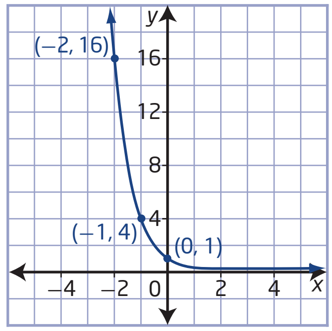


2. Consider the exponential function $y = 0.3^x$.

a) Make a table of values and sketch the graph of the function.

b) Identify the domain, range, intercepts, and intervals of increase or decrease, as well as any asymptotes.

3. What exponential function in the form $y = c^x$ is represented by the graph shown?



4. The value, v, of a dollar invested for t years at an annual interest rate of 3.25% is given by $v = 1.0325^t$.

a) Explain why the base of the exponential function is 1.0325.

b) What will be the value of \$1 if it is invested for 10 years?

c) How long will it take for the value of the dollar invested to reach \$2?

7.2 Transformations of Exponential Functions, page 346–357

5. The graph of $y = 4^x$ is transformed to obtain the graph of $y = -2(4)^{3(x-1)} + 2$.

a) What are the parameters and corresponding transformations?

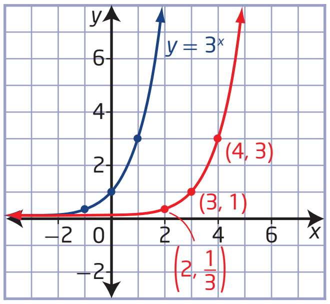
b) Copy and complete the table.

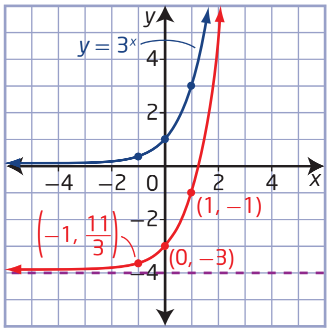
Transformation	Parameter Value	Function Equation
horizontal stretch		
vertical stretch		
translation left/right		
translation up/down		

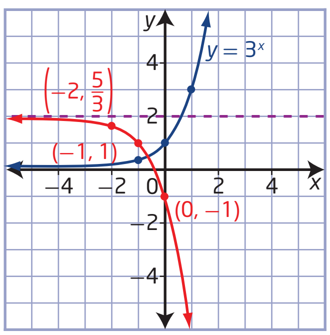
c) Sketch the graph of $y = -2(4)^{3(x-1)} + 2$.

d) Identify the domain, range, equation of the horizontal asymptote, and any intercepts for the function $y = -2(4)^{3(x-1)} + 2$.

6. Identify the transformation(s) used in each case to transform the base function $y = 3^x$.

a) 

b) 

c) 

7. Write the equation of the function that results from each set of transformations, and then sketch the graph of the function.

a) $f(x) = 5^x$ is stretched vertically by a factor of 4, stretched horizontally by a factor of $\frac{1}{2}$, reflected in the y-axis, and translated 1 unit up and 4 units to the left.

b) $g(x) = \left(\frac{1}{2}\right)^x$ is stretched horizontally by a factor of $\frac{1}{4}$, stretched vertically by a factor of 3, reflected in the x-axis, and translated 2 units to the right and 1 unit down.

8. The function $T = 190\left(\frac{1}{2}\right)^{\frac{1}{10}t}$ can be used to determine the length of time, t, in hours, that milk of a certain fat content will remain fresh. T is the storage temperature, in degrees Celsius.

a) Describe how each of the parameters in the function transforms the base function $T = \left(\frac{1}{2}\right)^t$.

b) Graph the transformed function.

c) What are the domain and range for this situation?

d) How long will milk keep fresh at 22 °C?

7.3 Solving Exponential Equations, pages 358–365

9. Write each as a power of 6.

a) 36

b) $\frac{1}{36}$

c) $\left(\sqrt[3]{216}\right)^5$

10. Solve each equation. Check your answers using graphing technology.

a) $3^{5x} = 27^{x-1}$

b) $\left(\frac{1}{8}\right)^{2x+1} = 32^{x-3}$

11. Solve for x. Round answers to two decimal places.

a) $3^{x-2} = 5^x$

b) $2^{x-2} = 3^{x+1}$

12. Nickel-65 (Ni-65) has a half-life of 2.5 h.

a) Write an exponential function to model this situation.

b) What fraction of a sample of Ni-65 will remain after 10 h?

c) How long will it take for a sample of Ni-65 to decay to $\frac{1}{1024}$ of its original mass?

Chapter 7 Practice Test

Multiple Choice

For #1 to #5, choose the best answer.

1. Consider the exponential functions $y = 2^x$, $y = \left(\frac{2}{3}\right)^x$, and $y = 7^x$. Which value of x results in the same y-value for each?

A -1

B 0

C 1

D There is no such value of x.

2. Which statement describes how to transform the function $y = 3^x$ into $y = 3^{\frac{1}{4}(x-5)} - 2$?

A stretch vertically by a factor of $\frac{1}{4}$ and translate 5 units to the left and 2 units up

B stretch horizontally by a factor of $\frac{1}{4}$ and translate 2 units to the right and 5 units down

C stretch horizontally by a factor of 4 and translate 5 units to the right and 2 units down

D stretch horizontally by a factor of 4 and translate 2 units to the left and 5 units up

3. An antique automobile was found to double in value every 10 years. If the current value is \$100 000, what was the value of the vehicle 20 years ago?

A \$50 000 **B** \$25 000

C \$12 500 **D** \$5000

4. What is $\frac{2^9}{(4^3)^2}$ expressed as a power of 2?

A 2^{-3}

B 2^3

C 2^1

D 2^{-1}

5. The intensity, I, in lumens, of light passing through the glass of a pair of sunglasses is given by the function $I(x) = I_0(0.8)^x$, where x is the thickness of the glass, in millimetres, and I_0 is the intensity of light entering the glasses. Approximately how thick should the glass be so that it will block 25% of the light entering the sunglasses?

A 0.7 mm

B 0.8 mm

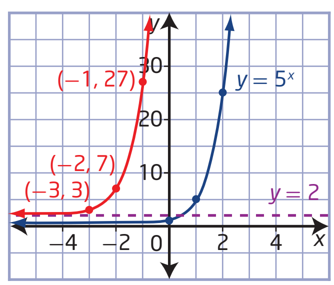
C 1.1 mm

D 1.3 mm

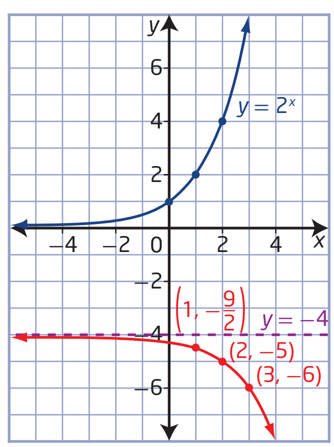
Short Answer

6. Determine the function that represents each transformed graph.

a)



b)



7. Sketch and label the graph of each exponential function.

a) $y = \frac{1}{2}(3)^x + 2$

b) $y = -2\left(\frac{3}{2}\right)^{x-1} - 2$

c) $y = 3^{2(x+3)} - 4$

8. Consider the function $g(x) = 2(3)^{x+3} - 4$.

a) Determine the base function for $g(x)$ and describe the transformations needed to transform the base function to $g(x)$.

b) Graph the function $g(x)$.

c) Identify the domain, the range, and the equation of the horizontal asymptote for $g(x)$.

9. Solve for x.

a) $3^{2x} = 9^{\frac{1}{2}(x-4)}$

b) $27^{x-4} = 9^{x+3}$

c) $1024^{2x-1} = 16^{x+4}$

10. Solve each equation using graphing technology. Round answers to one decimal place.

a) $3 = 1.12^x$

b) $2.7 = 0.3^{2x-1}$

Extended Response

11. According to a Statistics Canada report released in 2010, Saskatoon had the fastest-growing population in Canada, with an annual growth rate of 2.77%.

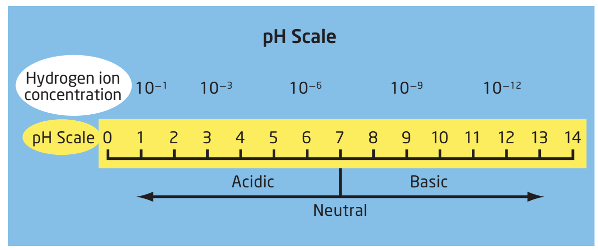
a) If the growth rate remained constant, by what factor would the population have been multiplied after 1 year?

b) What function could be used to model this situation?

c) What are the domain and range of the function for this situation?

d) At this rate, approximately how long would it take for Saskatoon's population to grow by 25%?

12. The measure of the acidity of a solution is called its pH. The pH of swimming pools needs to be checked regularly. This is done by measuring the concentration of hydrogen ions (H^+) in the water. The relationship between the hydrogen ion concentration, H, in moles per litre (mol/L), is $H(P) = \left(\frac{1}{10}\right)^P$, where P is the pH.



a) Sketch the graph of this function.

b) Water with a pH of less than 7.0 is acidic. What is the hydrogen ion concentration for a pH of 7.0?

c) Water in a swimming pool should have a pH of between 7.0 and 7.6. What is the equivalent range of hydrogen ion concentration?

13. Lucas is hoping to take a vacation after he finishes university. To do this, he estimates he needs $5000. Lucas is able to finish his last year of university with $3500 in an investment that pays 8.4% per year, compounded quarterly. How long will Lucas have to wait before he has enough money to take the vacation he wants?

14. A computer, originally purchased for $3000, depreciates in value according to the function $V(t) = 3000\left(\frac{1}{2}\right)^{\frac{t}{3}}$, where V is the value, in dollars, of the computer at any time, t, in years. Approximately how long will it take for the computer to be worth 10% of its purchase price?