The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2013 Euclid Contest

Wednesday, April 17, 2013
(in North America and South America)

Thursday, April 18, 2013
(outside of North America and South America)

Solutions

1. (a) The expression $\sqrt{113+x}$ is an integer whenever $113+x$ is a perfect square.
To find the smallest positive integer x for which $113+x$ is a perfect square, we find the smallest perfect square larger than 113.
Since $10^2 = 100$ and $11^2 = 121$, then this perfect square must be 121.
Therefore, $113 + x = 121$ or $x = 8$.

(b) The average of 3 and 11 is $\frac{1}{2}(3+11) = 7$. Thus $a = 7$.
Using this with the given information, we see that the average of 7 and b is 11.
Therefore, $\frac{1}{2}(7+b) = 11$ or $7 + b = 22$ and so $b = 15$.
(Alternatively, we could note that since 7 is 4 less than 11 (the average), then b must be 4 more than 11, so $b = 11 + 4 = 15$.)

(c) Let c be Charlie's age in years and b be Bella's age in years.
From the first sentence, $c = 30 + b$.
From the second sentence, $c = 6b$.
Combining these, we obtain $6b = 30 + b$ or $5b = 30$, and so $b = 6$.
Since $c = 30 + b$, then $c = 36$, and so Charlie's age is 36.

2. (a) Since $\dfrac{21}{x} = \dfrac{7}{y}$, then $21 = \dfrac{7x}{y}$ or $\dfrac{x}{y} = \dfrac{21}{7} = 3$.

(b) *Solution 1*
Since

$$\frac{1}{3} \approx 0.3333 \qquad \frac{1}{4} = 0.25 \qquad \frac{1}{5} = 0.2 \qquad \frac{1}{6} \approx 0.1667$$

then $\dfrac{1}{5} < 0.2013$ and $0.2013 < \dfrac{1}{4}$, so n must equal 4.
(We should note as well that $\dfrac{1}{n}$ decreases as n increases, so this is the only integer value of n that works.)

Solution 2
Since $\dfrac{1}{n+1} < 0.2013$, then $n + 1 > \dfrac{1}{0.2013}$ or $n > \dfrac{1}{0.2013} - 1 \approx 3.9677$.
Since $\dfrac{1}{n} > 0.2013$, then $n < \dfrac{1}{0.2013} \approx 4.9677$.
Since n is a positive integer

* that is smaller than a number that is approximately 4.9677, and
* that is larger than a number that is approximately 3.9677,

then $n = 4$.

(c) Since AH is perpendicular to BC, then the area of $\triangle ABC$ equals $\frac{1}{2}(BC)(AH)$.
Since we are told that this area equals 84 and $AH = 8$, then $84 = \frac{1}{2}(BC)(8)$ or $4 \cdot BC = 84$ or $BC = 21$.
Also, since $\triangle AHB$ is right-angled at H, then by the Pythagorean Theorem,

$$BH = \sqrt{AB^2 - AH^2} = \sqrt{10^2 - 8^2} = \sqrt{36} = 6$$

since $BH > 0$. (We could also have recognized two sides of a 6-8-10 right-angled triangle.)
Since $BC = 21$ and $BH = 6$, then $HC = BC - BH = 21 - 6 = 15$.

Since $\triangle AHC$ is right-angled at H, then by the Pythagorean Theorem,

$$AC = \sqrt{AH^2 + HC^2} = \sqrt{8^2 + 15^2} = \sqrt{289} = 17$$

since $AC > 0$.
Finally, the perimeter of $\triangle ABC$ equals $AB + BC + AC$ or $10 + 21 + 17$, which equals 48.

3. (a) The *parity* of an integer is whether it is even or odd.
Since the Fibonacci sequence begins $1, 1, 2, 3, 5, 8, 13, 21, \ldots$, then the parities of the first eight terms are Odd, Odd, Even, Odd, Odd, Even, Odd, Odd.
In the sequence, if x and y are consecutive terms, then the next term is $x + y$.
In general, suppose that x and y are integers.
If x is even and y is even, then $x + y$ is even. If x is even and y is odd, then $x + y$ is odd.
If x is odd and y is even, then $x + y$ is odd. If x is odd and y is odd, then $x + y$ is even.
Therefore, the parities of two consecutive terms x and y in the Fibonacci sequence determine the parity of the following term $x + y$.
Also, once there are two consecutive terms whose parities match the parities of two earlier consecutive terms in the sequence, then the parities will repeat in a cycle.
In particular, the parities of the fourth and fifth terms (Odd, Odd) are the same as the parities of the first and second terms (Odd, Odd).
Therefore, the parities in the sequence repeat the cycle Odd, Odd, Even.
This cycle has length 3.
Therefore, the 99th term in the Fibonacci sequence ends one of these cycles, since 99 is a multiple of 3.
In particular, the 99th term ends the 33rd cycle.
Each cycle contains two odd terms.
Therefore, the first 99 terms in the sequence include $2 \times 33 = 66$ odd terms.
Finally, the 100th term in the sequence begins a new cycle, so is odd.
Therefore, the first 100 terms include $66 + 1 = 67$ odd terms.

(b) Suppose that the first term in the given sequence is a and the common difference is d.
Then the first four terms are $a, a + d, a + 2d, a + 3d$.
From the given information, $a + (a + 2d) = 6$ and $(a + d) + (a + 3d) = 20$.
The first equation simplifies to $2a + 2d = 6$ or $a + d = 3$.
The second equation simplifies to $2a + 4d = 20$ or $a + 2d = 10$.
Therefore, $(a + 2d) - (a + d) = 10 - 3$ or $d = 7$.
Since $a + d = 3$ and $d = 7$, then $a = -4$.
Therefore, the tenth term in the sequence is $a + 9d = -4 + 9(7) = 59$.

4. (a) There are five odd digits: 1, 3, 5, 7, 9.
We consider the positive integers less than 1000 in three sets: those with one digit, those with two digits, and those with three digits.
There are 5 positive one-digit integers with one odd digit (namely 1, 3, 5, 7, 9).
Consider the two-digit positive integers with only odd digits.
Such an integer has the form XY where X and Y are digits.
There are five possibilities for each of X and Y (since each must be odd).
Therefore, there are $5 \times 5 = 25$ two-digit positive integers with only odd digits.
Consider the three-digit positive integers with only odd digits.
Such an integer has the form XYZ where X, Y and Z are digits.

There are five possibilities for each of X, Y and Z (since each must be odd).
Therefore, there are $5 \times 5 \times 5 = 125$ three-digit positive integers with only odd digits.
In total, there are $5 + 25 + 125 = 155$ positive integers less than 1000 with only odd digits.

(b) Combining the two terms on the right side of the second equation, we obtain $\dfrac{4}{7} = \dfrac{b+a}{ab}$.
Since $a + b = 16$, then $\dfrac{4}{7} = \dfrac{16}{ab}$ or $ab = \dfrac{16(7)}{4} = 28$.
Therefore, we have $a + b = 16$ and $ab = 28$.
From the first equation, $b = 16 - a$.
Substituting into the second equation, we obtain $a(16 - a) = 28$ or $16a - a^2 = 28$, which gives $a^2 - 16a + 28 = 0$.
Factoring, we obtain $(a - 14)(a - 2) = 0$.
Therefore, $a = 14$ or $a = 2$.
If $a = 14$, then $b = 16 - a = 2$.
If $a = 2$, then $b = 16 - a = 14$.
Therefore, the two solutions are $(a, b) = (14, 2), (2, 14)$.
(We note that since $\dfrac{1}{2} + \dfrac{1}{14} = \dfrac{7}{14} + \dfrac{1}{14} = \dfrac{8}{14} = \dfrac{4}{7}$, then both of these pairs are actually solutions to the original system of equations.)

5. (a) We make a table of the 36 possible combinations of rolls and the resulting sums:

	2	3	5	7	11	13
2	4	5	7	9	13	15
3	5	6	8	10	14	16
5	7	8	10	12	16	18
7	9	10	12	14	18	20
11	13	14	16	18	22	24
13	15	16	18	20	24	26

Of the 36 entries in the table, 6 are prime numbers (two entries each of 5, 7 and 13).
Therefore, the probability that the sum is a prime number is $\frac{6}{36}$ or $\frac{1}{6}$.
(Note that each sum is at least 4 and so must be odd to be prime. Since odd plus odd equals even, then the only possibilities that really need to be checked are even plus odd and odd plus even (that is, the first row and first column of the table).)

(b) First, we find the coordinates of V.
To do this, we use the given equation for the parabola and complete the square:

$$y = -x^2+4x+1 = -(x^2-4x-1) = -(x^2-4x+2^2-2^2-1) = -((x-2)^2-5) = -(x-2)^2+5$$

Therefore, the coordinates of the vertex V are $(2, 5)$.
Next, we find the coordinates of A and B.
Note that A and B are the points of intersection of the line with equation $y = -x + 1$ and the parabola with equation $y = -x^2 + 4x + 1$.
We equate y-values to obtain $-x + 1 = -x^2 + 4x + 1$ or $x^2 - 5x = 0$ or $x(x - 5) = 0$.
Therefore, $x = 0$ or $x = 5$.
If $x = 0$, then $y = -x + 1 = 1$, and so A (which is on the y-axis) has coordinates $(0, 1)$.
If $x = 5$, then $y = -x + 1 = -4$, and so B has coordinates $(5, -4)$.

We now have the points $V(2,5)$, $A(0,1)$, $B(5,-4)$.
This gives

$$\begin{aligned} AV^2 &= (0-2)^2+(1-5)^2 = 20 \\ BV^2 &= (5-2)^2+(-4-5)^2 = 90 \\ AB^2 &= (0-5)^2+(1-(-4))^2 = 50 \end{aligned}$$

and so $AV^2+BV^2-AB^2 = 20+90-50 = 60$.

6. (a) Since ABC is a quarter of a circular pizza with centre A and radius 20 cm, then $AC = AB = 20$ cm.
We are also told that $\angle CAB = 90^\circ$ (one-quarter of 360°).
Since $\angle CAB = 90^\circ$ and A, B and C are all on the circumference of the circle, then CB is a diameter of the pan. (This is a property of circles: if X, Y and Z are three points on a circle with $\angle ZXY = 90^\circ$, then YZ must be a diameter of the circle.)
Since $\triangle CAB$ is right-angled and isosceles, then $CB = \sqrt{2}AC = 20\sqrt{2}$ cm.
Therefore, the radius of the circular plate is $\frac{1}{2}CB$ or $10\sqrt{2}$ cm.
Thus, the area of the circular pan is $\pi(10\sqrt{2}\text{ cm})^2 = 200\pi\text{ cm}^2$.
The area of the slice of pizza is one-quarter of the area of a circle with radius 20 cm, or $\frac{1}{4}\pi(20\text{ cm})^2 = 100\pi\text{ cm}^2$.
Finally, the fraction of the pan that is covered is the area of the slice of pizza divided by the area of the pan, or $\dfrac{100\pi\text{ cm}^2}{200\pi\text{ cm}^2} = \dfrac{1}{2}$.

(b) Suppose that the length of AF is x m.
Since the length of AB is 8 m, then the length of FB is $(8-x)$ m.
Since $\triangle MAF$ is right-angled and has an angle of 60°, then it is 30°-60°-90° triangle.
Therefore, $MF = \sqrt{3}AF$, since MF is opposite the 60° angle and AF is opposite the 30° angle.
Thus, $MF = \sqrt{3}x$ m.
Since $MP = 2$ m, then $PF = MF - MP = (\sqrt{3}x - 2)$ m.
We can now look at $\triangle BFP$ which is right-angled at F.
We have

$$\tan\theta = \frac{PF}{FB} = \frac{(\sqrt{3}x-2)\text{ m}}{(8-x)\text{ m}} = \frac{\sqrt{3}x-2}{8-x}$$

Therefore, $(8-x)\tan\theta = \sqrt{3}x - 2$ or $8\tan\theta + 2 = \sqrt{3}x + (\tan\theta)x$.
This gives $8\tan\theta + 2 = x(\sqrt{3}+\tan\theta)$ or $x = \dfrac{8\tan\theta+2}{\tan\theta+\sqrt{3}}$.
Finally, $MF = \sqrt{3}x = \dfrac{8\sqrt{3}\tan\theta + 2\sqrt{3}}{\tan\theta+\sqrt{3}}$ m.

7. (a) Beginning with the given equation, we have

$$\begin{aligned}
\frac{1}{\cos x} - \tan x &= 3 \\
\frac{1}{\cos x} - \frac{\sin x}{\cos x} &= 3 \\
1 - \sin x &= 3\cos x \qquad \text{(since } \cos x \neq 0\text{)} \\
(1-\sin x)^2 &= 9\cos^2 x \qquad \text{(squaring both sides)} \\
1 - 2\sin x + \sin^2 x &= 9(1-\sin^2 x) \\
10\sin^2 x - 2\sin x - 8 &= 0 \\
5\sin^2 x - \sin x - 4 &= 0 \\
(5\sin x + 4)(\sin x - 1) &= 0
\end{aligned}$$

Therefore, $\sin x = -\frac{4}{5}$ or $\sin x = 1$.
If $\sin x = 1$, then $\cos x = 0$ and $\tan x$ is undefined, which is inadmissible in the original equation.
Therefore, $\sin x = -\frac{4}{5}$.
(We can check that if $\sin x = -\frac{4}{5}$, then $\cos x = \pm\frac{3}{5}$ and the possibility that $\cos x = \frac{3}{5}$ satisfies the original equation, since in this case $\frac{1}{\cos x} = \frac{5}{3}$ and $\tan x = -\frac{4}{3}$ and the difference between these fractions is 3.)

(b) Since $f(x) = ax+b$, we can determine an expression for $g(x) = f^{-1}(x)$ by letting $y = f(x)$ and to obtain $y = ax + b$. We then interchange x and y to obtain $x = ay + b$ which we solve for y to obtain $ay = x - b$ or $y = \frac{x}{a} - \frac{b}{a}$.
Therefore, $f^{-1}(x) = \frac{x}{a} - \frac{b}{a}$.
Note that $a \neq 0$. (This makes sense since the function $f(x) = b$ has a graph which is a horizontal line, and so cannot be invertible.)
Therefore, the equation $f(x) - g(x) = 44$ becomes $(ax + b) - \left(\frac{x}{a} - \frac{b}{a}\right) = 44$ or $\left(a - \frac{1}{a}\right)x + \left(b + \frac{b}{a}\right) = 44 = 0x + 44$, and this equation is true for all x.
We can proceed in two ways.

<u>Method #1: Comparing coefficients</u>
Since the equation

$$\left(a - \frac{1}{a}\right)x + \left(b + \frac{b}{a}\right) = 0x + 44$$

is true for all x, then the coefficients of the linear expression on the left side must match the coefficients of the linear expression on the right side.
Therefore, $a - \frac{1}{a} = 0$ and $b + \frac{b}{a} = 44$.
From the first of these equations, we obtain $a = \frac{1}{a}$ or $a^2 = 1$, which gives $a = 1$ or $a = -1$.
If $a = 1$, the equation $b + \frac{b}{a} = 44$ becomes $b + b = 44$, which gives $b = 22$.

If $a = -1$, the equation $b + \frac{b}{a} = 44$ becomes $b - b = 44$, which is not possible.
Therefore, we must have $a = 1$ and $b = 22$, and so $f(x) = x + 22$.

Method #2: Trying specific values for x
Since the equation

$$\left(a - \frac{1}{a}\right)x + \left(b + \frac{b}{a}\right) = 0x + 44$$

is true for all values of x, then it must be true for any specific values of x that we choose.
Choosing $x = 0$, we obtain $0 + \left(b + \frac{b}{a}\right) = 44$ or $b + \frac{b}{a} = 44$.
Choosing $x = b$, we obtain $\left(a - \frac{1}{a}\right)b + \left(b + \frac{b}{a}\right) = 44$ or $ab + b = 44$.
We can rearrange the first of these equations to get $\frac{ab + b}{a} = 44$.
Using the second equation, we obtain $\frac{44}{a} = 44$ or $a = 1$.
Since $a = 1$, then $ab + b = 44$ gives $2b = 44$ or $b = 22$.
Thus, $f(x) = x + 22$.

In summary, the only linear function f for which the given equation is true for all x is $f(x) = x + 22$.

8. (a) First, we factor the left side of the given equation to obtain $a(a^2 + 2b) = 2013$.
Next, we factor the integer 2013 as $2013 = 3 \times 671 = 3 \times 11 \times 61$. Note that each of 3, 11 and 61 is prime, so we can factor 2013 no further. (We can find the factors of 3 and 11 using tests for divisibility by 3 and 11, or by systematic trial and error.)
Since $2013 = 3 \times 11 \times 61$, then the positive divisors of 2013 are

$$1, 3, 11, 33, 61, 183, 671, 2013$$

Since a and b are positive integers, then a and $a^2 + 2b$ are both positive integers.
Since a and b are positive integers, then $a^2 \geq a$ and $2b > 0$, so $a^2 + 2b > a$.
Since $a(a^2 + 2b) = 2013$, then a and $a^2 + 2b$ must be a divisor pair of 2013 (that is, a pair of positive integers whose product is 2013) with $a < a^2 + 2b$.
We make a table of the possibilities:

a	$a^2 + 2b$	$2b$	b
1	2013	2012	1006
3	671	662	331
11	183	62	31
33	61	-1028	N/A

Note that the last case is not possible, since b must be positive.
Therefore, the three pairs of positive integers that satisfy the equation are $(1, 1006)$, $(3, 331)$, $(11, 31)$.
(We can verify by substitution that each is a solution of the original equation.)

(b) *Solution 1*

We successively manipulate the given equation to produce equivalent equations:

$$\begin{aligned}
\log_2(2^{x-1}+3^{x+1}) &= 2x - \log_2(3^x) \\
\log_2(2^{x-1}+3^{x+1}) + \log_2(3^x) &= 2x \\
\log_2((2^{x-1}+3^{x+1})3^x) &= 2x \qquad \text{(using } \log_2 A + \log_2 B = \log_2 AB\text{)} \\
(2^{x-1}+3^{x+1})3^x &= 2^{2x} \qquad \text{(exponentiating both sides)} \\
2^{-1}2^x3^x + 3^13^x3^x &= 2^{2x} \\
\tfrac{1}{2}\cdot 2^x3^x + 3\cdot 3^{2x} &= 2^{2x} \\
2^x3^x + 6\cdot 3^{2x} &= 2\cdot 2^{2x} \qquad \text{(multiplying by 2)} \\
2^x3^x + 6\cdot(3^x)^2 &= 2\cdot(2^x)^2
\end{aligned}$$

Next, we make the substitution $a = 2^x$ and $b = 3^x$.
This gives $ab + 6b^2 = 2a^2$ or $2a^2 - ab - 6b^2 = 0$.
Factoring, we obtain $(a-2b)(2a+3b) = 0$.
Therefore, $a = 2b$ or $2a = -3b$.
Since $a > 0$ and $b > 0$, then $a = 2b$ which gives $2^x = 2\cdot 3^x$.
Taking logs of both sides, we obtain $x\log 2 = \log 2 + x\log 3$ and so $x(\log 2 - \log 3) = \log 2$ or $x = \dfrac{\log 2}{\log 2 - \log 3}$.

Solution 2

We successively manipulate the given equation to produce equivalent equations:

$$\begin{aligned}
\log_2(2^{x-1}+3^{x+1}) &= 2x - \log_2(3^x) \\
\log_2(2^{x-1}+3^{x+1}) + \log_2(3^x) &= 2x \\
\log_2((2^{x-1}+3^{x+1})3^x) &= 2x \qquad \text{(using } \log_2 A + \log_2 B = \log_2 AB\text{)} \\
(2^{x-1}+3^{x+1})3^x &= 2^{2x} \qquad \text{(exponentiating both sides)} \\
2^{-1}2^x3^x + 3^13^x3^x &= 2^{2x} \\
\tfrac{1}{2}\cdot 2^x3^x + 3\cdot 3^{2x} &= 2^{2x} \\
2^x3^x + 6\cdot 3^{2x} &= 2\cdot 2^{2x} \qquad \text{(multiplying by 2)} \\
2^x3^x2^{-2x} + 6\cdot 3^{2x}2^{-2x} &= 2 \qquad \text{(dividing both sides by } 2^{2x} \neq 0\text{)} \\
2^{-x}3^x + 6\cdot 3^{2x}2^{-2x} &= 2 \\
\left(\tfrac{3}{2}\right)^x + 6\left(\tfrac{3}{2}\right)^{2x} &= 2
\end{aligned}$$

Next, we make the substitution $t = \left(\frac{3}{2}\right)^x$, noting that $\left(\frac{3}{2}\right)^{2x} = \left(\left(\frac{3}{2}\right)^x\right)^2 = t^2$.
Thus, we obtain the equivalent equations

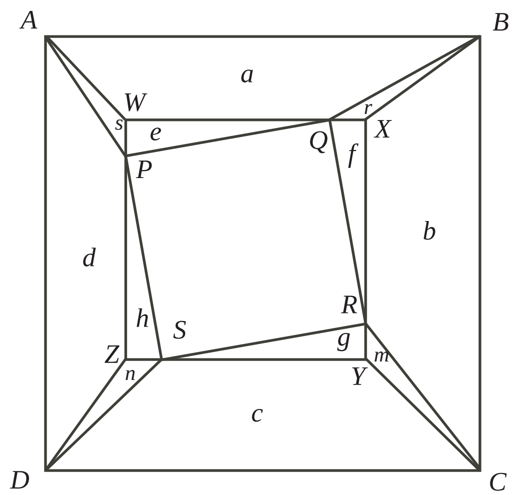
$$\begin{aligned}
t + 6t^2 &= 2 \\
6t^2 + t - 2 &= 0 \\
(3t+2)(2t-1) &= 0
\end{aligned}$$

Therefore, $t = -\frac{2}{3}$ or $t = \frac{1}{2}$.
Since $t = \left(\frac{3}{2}\right)^x > 0$, then we must have $t = \left(\frac{3}{2}\right)^x = \frac{1}{2}$.
Thus,

$$x = \log_{3/2}(1/2) = \frac{\log(1/2)}{\log(3/2)} = \frac{\log 1 - \log 2}{\log 3 - \log 2} = \frac{-\log 2}{\log 3 - \log 2} = \frac{\log 2}{\log 2 - \log 3}$$

9. (a) Suppose that the parallel line segments EF and WX are a distance of x apart.
This means that the height of trapezoid $EFXW$ is x.
Since the side length of square $EFGH$ is 10 and the side length of square $WXYZ$ is 6, then the distance between parallel line segments ZY and HG is $10-6-x$ or $4-x$.
Recall that the area of a trapezoid equals one-half times its height times the sum of the lengths of the parallel sides.
Thus, the area of trapezoid $EFXW$ is $\frac{1}{2}x(EF+WX)=\frac{1}{2}x(10+6)=8x$.
Also, the area of trapezoid $GHZY$ is $\frac{1}{2}(4-x)(HG+ZY)=\frac{1}{2}(4-x)(10+6)=32-8x$.
Therefore, the sum of the areas of trapezoids $EFXW$ and $GHZY$ is $8x+(32-8x)=32$.
This sum is a constant and does not depend on the position of the inner square within the outer square, as required.

(b) We begin by "boxing in" square $PQRS$ by drawing horizontal and vertical lines through its vertices to form rectangle $WXYZ$, as shown. (Because the four quadrilaterals $ABQP$, $BCRQ$, $CDSR$, and $DAPS$ are convex, there will not be any configurations that look substantially different from this the diagram below.) We also label the various areas.



Since WX is parallel to AB, then quadrilateral $ABXW$ is a trapezoid. Similarly, quadrilaterals $BCYX$, $CDZY$, and $DAWZ$ are trapezoids.
We use the notation $|ABQP|$ to denote the area of quadrilateral $ABQP$, and similar notation for other areas.
Suppose that the side length of square $ABCD$ is x and the side length of square $PQRS$ is y.
Also, we let $\angle WPQ=\theta$.
Since each of $\triangle WPQ$, $\triangle XQR$, $\triangle YRS$, and $\triangle ZSP$ is right-angled and each of the four angles of square $PQRS$ is 90°, then $\angle WPQ=\angle XQR=\angle YRS=\angle ZSP=\theta$. This is because, for example,

$$\angle XQR=180^\circ-\angle PQR-\angle WQP=90^\circ-(180^\circ-\angle PWQ-\angle WPQ)=90^\circ-(90^\circ-\theta)=\theta$$

This fact, together with the fact that $PQ=QR=RS=SP=y$, allows us to conclude that the four triangles $\triangle WPQ$, $\triangle XQR$, $\triangle YRS$, and $\triangle ZSP$ are congruent.
In particular, this tells us

* the four areas labelled e, f, g and h are equal (that is, $e=f=g=h$),
* $PZ=QW=RX=SY=y\sin\theta$, and
* $WP=XQ=YR=ZS=y\cos\theta$.

Combining these last two facts tells us that $WZ = XW = YX = ZY$, since, for example, $WZ = WP + PZ = ZS + SY = ZY$. In other words, $WXYZ$ is a square, with side length z, say.

Next, we show that $(a+r)+(c+n)$ is equal to $(b+m)+(d+s)$.
Note that the sum of these two quantities is the total area between square $ABCD$ and square $WXYZ$, so equals $x^2 - z^2$.
Thus, to show that the quantities are equal, it is enough to show that $(a+r)+(c+n)$ equals $\frac{1}{2}(x^2 - z^2)$.
Let the height of trapezoid $ABXW$ be k and the height of trapezoid $ZYCD$ be l.
Then $|ABXW| = a + r = \frac{1}{2}k(AB + WX) = \frac{1}{2}k(x+z)$.
Also, $|ZYCD| = c + n = \frac{1}{2}l(DC + ZY) = \frac{1}{2}l(x+z)$.
Since AB, WX, ZY, and DC are parallel, then the sum of the heights of trapezoid $ABXW$, square $WXYZ$, and trapezoid $ZYCD$ equals the height of square $ABCD$, so $k + z + l = x$, or $k + l = x - z$.
Therefore,

$$(a+r)+(c+n) = \tfrac{1}{2}k(x+z) + \tfrac{1}{2}l(x+z) = \tfrac{1}{2}(x+z)(k+l) = \tfrac{1}{2}(x+z)(x-z) = \tfrac{1}{2}(x^2-z^2)$$

as required.
Therefore, $(a+r)+(c+n) = (b+m)+(d+s)$. We label this equation $(*)$.

Next, we show that $r + n = m + s$.
Note that $r = |\triangle QXB|$. This triangle can be viewed as having base QX and height equal to the height of trapezoid $ABXW$, or k.
Thus, $r = \frac{1}{2}(y\cos\theta)k$.
Note that $n = |\triangle SZD|$. This triangle can be viewed as having base SZ and height equal to the height of trapezoid $ZYCD$, or l.
Thus, $n = \frac{1}{2}(y\cos\theta)l$.
Combining these facts, we obtain

$$n + r = \tfrac{1}{2}(y\cos\theta)k + \tfrac{1}{2}(y\cos\theta)l = \tfrac{1}{2}y\cos\theta(k+l) = \tfrac{1}{2}y\cos\theta(x-z)$$

We note that this sum depends only on the side lengths of the squares and the angle of rotation of the inner square, so is independent of the position of the inner square within the outer square.
This means that we can repeat this analysis to obtain the same expression for $m + s$.
Therefore, $n + r = m + s$. We label this equation $(**)$.

We subtract $(*) - (**)$ to obtain $a + c = b + d$.
Finally, we can combine all of this information:

$$\begin{aligned}
&(|ABQP| + |CDSR|) - (|BCRQ| + |APSD|) \\
&\quad = (a+e+s+c+g+m) - (b+f+r+d+h+n) \\
&\quad = ((a+c)-(b+d)) + ((m+s)-(n+r)) + ((e+g)-(f+h)) \\
&\quad = 0 + 0 + 0
\end{aligned}$$

since $a + c = b + d$ and $n + r = m + s$ and $e = f = g = h$.
Therefore, $|ABQP| + |CDSR| = |BCRQ| + |APSD|$, as required.

10. In each part, we use "partition" to mean "multiplicative partition". We also call the numbers being multiplied together in a given partition the "parts" of the partition.

(a) We determine the multiplicative partitions of 64 by considering the number of parts in the various partitions. Note that 64 is a power of 2 so any divisor of 64 is also a power of 2. In each partition, since the order of parts is not important, we list the parts in increasing order to make it easier to systematically find all of these.

* One part. There is one possibility: 64.
* Two parts. There are three possibilities: $64 = 2 \times 32 = 4 \times 16 = 8 \times 8$.
* Three parts. We start with the smallest possible first and second parts. We keep the first part fixed while adjusting the second and third parts. We then increase the first part and repeat.
  We get: $64 = 2 \times 2 \times 16 = 2 \times 4 \times 8 = 4 \times 4 \times 4$.
* Four parts. A partition of 64 with four parts must include at least two 2s, since if it didn't, it would include at least three parts that are at least 4, and so would be too large. With two 2s, the remaining two parts have a product of 16.
  We get: $64 = 2 \times 2 \times 2 \times 8 = 2 \times 2 \times 4 \times 4$.
* Five parts. A partition of 64 with five parts must include at least three 2s, since if it didn't, it would include at least three parts that are at least 4, and so would be too large. With three 2s, the remaining two parts have a product of 8.
  We get: $64 = 2 \times 2 \times 2 \times 2 \times 4$.
* Six parts. Since $64 = 2^6$, there is only one possibility: $64 = 2 \times 2 \times 2 \times 2 \times 2 \times 2$.

Therefore, $P(64) = 1 + 3 + 3 + 2 + 1 + 1 = 11$.

(b) First, we note that $1000 = 10^3 = (2 \cdot 5)^3 = 2^3 5^3$.
We calculate the value of $P(p^3q^3)$ for two distinct prime numbers p and q. It will turn out that this value does not depend on p and q. This value will be the value of $P(1000)$, since 1000 has this form of prime factorization.

Let $n = p^3q^3$ for distinct prime numbers p and q.
The integer n has three prime factors equal to p.
In a given partition, these can be all together in one part (as p^3), can be split between two different parts (as p and p^2), or can be split between three different parts (as p, p and p). There are no other ways to divide up three divisors of p.
Similarly, n has three prime factors equal to q which can be divided in similar ways.
We determine $P(p^3q^3)$ by considering the possible combination of the number of parts divisible by p and the number of parts divisible by q and counting partitions in each case. In other words, we complete the following table:

		Number of parts divisible by p		
		1	2	3
Number of parts divisible by q	1			
	2			
	3			

We note that the table is symmetric, since the factors of p and q are interchangeable.
We proceed to consider cases, considering only those on the top left to bottom right diagonal and and those below this diagonal in the table.

Case 1: One part divisible by p, one part divisible by q

The partition must be p^3q^3 (n itself) or $p^3 \times q^3$.
There are two partitions in this case.

Case 2: One part divisible by p, two parts divisible by q

The three factors of p occur together as p^3. The three factors of q occur as q and q^2.
The p^3 can occur in one of the parts divisible by q or not.
This gives partitions $p^3 \times q \times q^2$ and $p^3q \times q^2$ and $q \times p^3q^2$.
There are three partitions in this case. Similarly, there are three partitions with one part divisible by q and two parts divisible by p.

Case 3: One part divisible by p, three parts divisible by q

The three factors of p occur together as p^3. The three factors of q occur as q, q and q.
The p^3 can occur in one of the parts divisible by q or not.
This gives partitions $p^3 \times q \times q \times q$ and $p^3q \times q \times q$.
(Note that the three divisors of q are interchangeable so p^3 only needs to be placed with one of them.)
There are two partitions in this case. Similarly, there are two partitions with one part divisible by q and three parts divisible by p.

Case 4: Two parts divisible by p, two parts divisible by q

The three factors of p occur as p and p^2. The three factors of q occur as q and q^2.
Each of p and p^2 can occur in one of the parts divisible by q or not.
If no part is a multiple of both p and q, we have one partition: $p \times p^2 \times q \times q^2$.
If one part is a multiple of both p and q, there are two choices for which power of p to include in this part and two choices for which power of q to include. (There is no choice for the remaining parts.) Thus, there are $2 \times 2 = 4$ such partitions:

$$p^2q^2 \times p \times q \qquad pq^2 \times p^2 \times q \qquad p^2q \times p \times q^2 \qquad pq \times p^2 \times q^2$$

If two parts are a multiple of both p and q, there are two ways to choose the power of p in the part containing just q, so there are two such partitions: $pq \times p^2q^2$ and $p^2q \times pq^2$.
There are seven partitions in this case.

Case 5: Two parts divisible by p, three parts divisible by q

The three factors of p occur as p and p^2. The three factors of q occur as q, q and q.
Each of p and p^2 can occur in one of the parts divisible by q or not.
If no part is a multiple of both p and q, we have one partition: $p \times p^2 \times q \times q \times q$.
If one part is a multiple of both p and q, there are two choices for which power of p to include in this part (since all powers of q are identical).
Thus, there are 2 such partitions: $p^2q \times p \times q \times q$ and $pq \times p^2 \times q \times q$.
If two parts are a multiple of both p and q, there is one partition, since all of the powers of q are identical: $pq \times p^2q \times q$.
There are four partitions in this case. Similarly, there are four partitions with two parts divisible by q and three parts divisible by p.

Case 6: Three parts divisible by p, three parts divisible by q

The three factors of p as p, p and p. The three factors of q appear as q, q and q.
Here, the number of parts in the partition that are multiples of both p and q can be 0, 1, 2 or 3. Since all of the powers of p and q are identical, the partitions are completely determined by this and are

$$p \times p \times p \times q \times q \times q \qquad p \times p \times pq \times q \times q \qquad p \times pq \times pq \times q \qquad pq \times pq \times pq$$

There are four partitions in this case.

Finally, we complete the table:

Number of parts divisible by q \ Number of parts divisible by p	1	2	3
1	2	3	2
2	3	7	4
3	2	4	4

Adding the entries in the table, we obtain $P(p^3q^3) = 31$.
Thus, $P(1000) = 31$.

(c) As in (b), the value of $P(n)$ depends only on the structure of the prime factorization of n, not on the actual primes in the factorization.
Therefore, $P(4 \times 5^m) = P(2^2 \times 5^m) = P(p^2q^m)$ for any distinct primes p and q.
Therefore, $P(4 \times 5^m) = P(p^2q^m) = P(5^2 \times 2^m) = P(25 \times 2^m)$.
We count the number of multiplicative partitions of $N = 5^2 \times 2^m$ by considering the placement of the 2s and 5s among the parts of the partitions.
Since N has only two factors of 5, these can occur in the same part, or in different parts.
Note that every factor of N is a product is of the form 5^j2^k for some integers j and k with $0 \le j \le 2$ and $0 \le k \le m$.

We first count the number of partitions where the two factors of 5 occur in the same part.
Consider one such partition.
In this partition, the part containing the two 5s will be of the form 5^22^k for some integer k with $0 \le k \le m$.
Thus, this partition will be of the form $5^22^k \times \mathcal{P}$, where $\mathcal{P}$ is a partition of 2^{m-k} (the remaining factors in N).
Since the order of parts does not matter, there are $P(2^{m-k})$ such partitions $\mathcal{P}$, and so this number of partitions of N of this form.
Since k ranges from 0 to m, then the number of partitions where the two 5s occur in the same part equals

$$P(2^m) + P(2^{m-1}) + \cdots + P(2^1) + P(2^0)$$

Next, we count the number of partitions where the two factors of 5 occur in different parts.
Consider one such partition.
In this partition, the parts containing the two 5s will be of the form 5×2^a and 5×2^b for some integers a and b with $0 \le a, b \le m$ and $a + b \le m$.
Since the order of the parts within a partition does not matter, we can restrict a and b further by requiring that $0 \le a \le b \le m$ and $a + b \le m$ to avoid double-counting partitions.
Thus, this partition will be of the form $(5 \times 2^a) \times (5 \times 2^b) \times \mathcal{P}$, where $\mathcal{P}$ is a partition of 2^{m-a-b} (the remaining factors in N).
Since the order of parts does not matter, there are $P(2^{m-a-b})$ such partitions $\mathcal{P}$, and so this number of partitions of N of this form.

To determine the total number of partitions in this case, we need to add up $P(2^{m-a-b})$ over all possible pairs (a, b) satisfying $0 \le a \le b \le m$ and $a + b \le m$.
To do this, we focus on the possible values of $s = a + b$ and count the number of pairs (a, b) that give this sum.

If $s = a + b = 0$, there is one pair (a, b), namely $(a, b) = (0, 0)$.
If $s = a + b = 1$, there is one pair (a, b), namely $(a, b) = (0, 1)$.
If $s = a + b = 2$, there are two pairs (a, b), namely $(a, b) = (0, 2), (1, 1)$.
In general, if s is even, then $\frac{1}{2}s$ is an integer and so there are $(\frac{1}{2}s + 1)$ pairs (a, b), namely

$$(0, s), (1, s - 1), (2, s - 2), \ldots, (\tfrac{1}{2}s - 1, \tfrac{1}{2}s + 1), (\tfrac{1}{2}s, \tfrac{1}{2}s)$$

Any larger value of a would give a value of b smaller than a.
In general, if s is odd, then $\frac{1}{2}s - \frac{1}{2}$ is an integer and so there are $(\frac{1}{2}s - \frac{1}{2}) + 1 = (\frac{1}{2}s + \frac{1}{2})$ pairs (a, b), namely

$$(0, s), (1, s - 1), (2, s - 2), \ldots, (\tfrac{1}{2}s - \tfrac{3}{2}, \tfrac{1}{2}s + \tfrac{3}{2}), (\tfrac{1}{2}s - \tfrac{1}{2}, \tfrac{1}{2}s + \tfrac{1}{2})$$

Any larger value of a would give a value of b smaller than a.
To summarize, if $s = a + b$ is even, there are $(\frac{1}{2}s + 1)$ pairs (a, b) and if $s = a + b$ is odd, there are $(\frac{1}{2}s + \frac{1}{2})$ pairs (a, b).
Thus, as s increases from 0, the number of pairs (a, b) gives the sequence $1, 1, 2, 2, 3, 3, \ldots$.
The number in this sequence corresponding to the value of $a+b$ gives the number of times that $P(2^{m-a-b})$ should be included in the count of the total number of partitions in this case.
In other words, if $a + b = 0$, there are $1 \times P(2^m)$ partitions, if $a + b = 1$, there are $1 \times P(2^{m-1})$ partitions, if $a + b = 2$, there are $2 \times P(2^{m-2})$ partitions, etc.
We can rewrite this more compactly to say that for a given s, the number of pairs (a, b) is $\left\lfloor \dfrac{s+2}{2} \right\rfloor$ (where $\lfloor x \rfloor$ is the greatest integer less than or equal to x) and so the number of partitions is $\left\lfloor \dfrac{s+2}{2} \right\rfloor \times P(2^{m-s})$.
Therefore, the total number of partitions of N in this case is

$$1 \times P(2^m) + 1 \times P(2^{m-1}) + 2 \times P(2^{m-2}) + 2 \times P(2^{m-3}) + \cdots + \left\lfloor \frac{s+2}{2} \right\rfloor \times P(2^{m-s}) + \cdots$$
$$+ \left\lfloor \frac{m}{2} \right\rfloor \times P(2^1) + \left\lfloor \frac{m+2}{2} \right\rfloor \times P(2^0)$$

Combining the two cases and adding the corresponding expressions for the number of partitions, we obtain that the total number of partitions is

$$2 \times P(2^m) + 2 \times P(2^{m-1}) + 3 \times P(2^{m-2}) + 3 \times P(2^{m-3}) + \cdots + \left(1 + \left\lfloor \frac{s+2}{2} \right\rfloor\right) \times P(2^{m-s}) + \cdots$$
$$+ \left(1 + \left\lfloor \frac{m+1}{2} \right\rfloor\right) \times P(2^1) + \left(1 + \left\lfloor \frac{m+2}{2} \right\rfloor\right) \times P(2^0)$$

and so the desired sequence is

$$\begin{aligned} a_0 &= 2 \\ a_1 &= 2 \\ a_2 &= 3 \\ a_3 &= 3 \\ &\vdots \\ a_s &= 1 + \left\lfloor \frac{s+2}{2} \right\rfloor \\ &\vdots \end{aligned}$$